CMSC 28100

Introduction to Complexity Theory

Spring 2025
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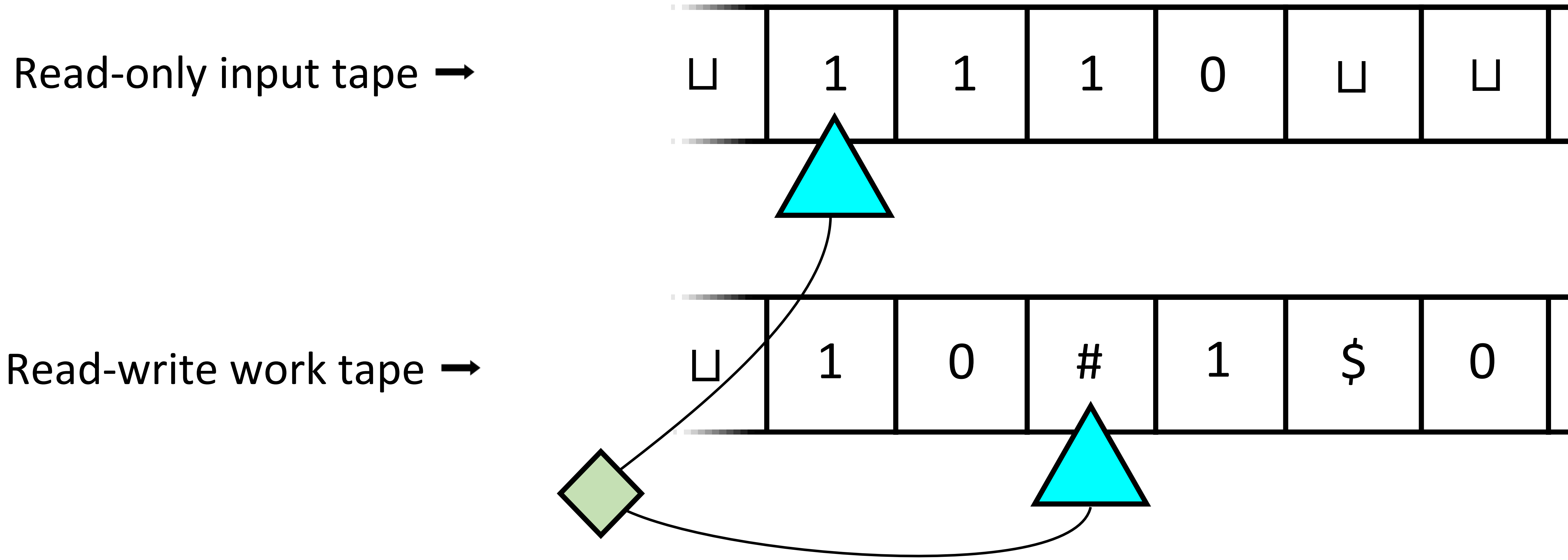
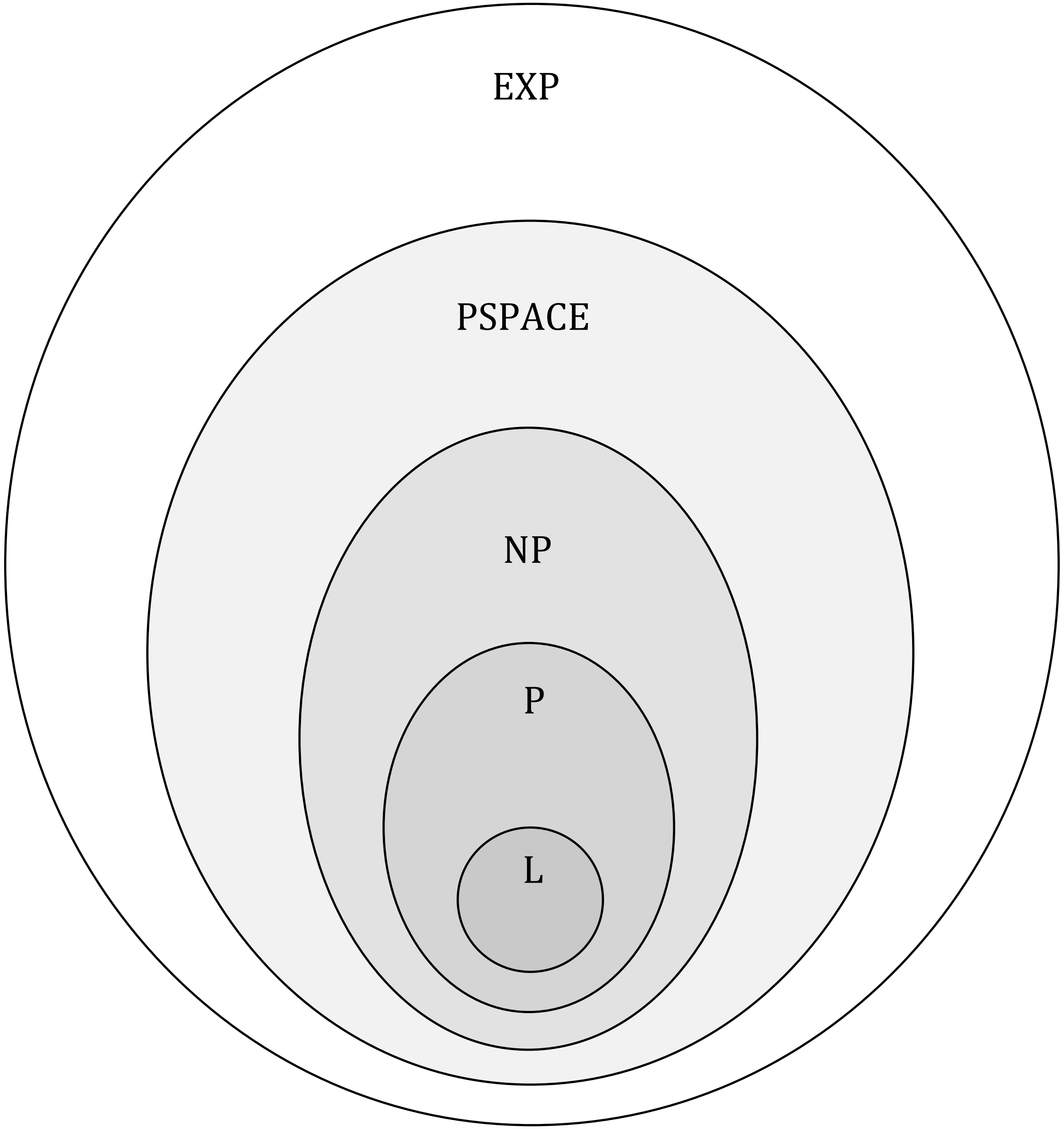
Sublinear-space computation



The complexity class $\mathbf{L}$

- $\mathbf{L}$ is the set of languages that can be decided using $O(\log n)$ cells of the work tape


EXP
PSPACE
NP
P
L

Nondeterministic log space computation

- We define NL to be the class of languages that can be decided by a nondeterministic log-space Turing machine
- Equivalently: NL is the class of languages for which membership can be verified in logarithmic space – with the extra requirement that the verifier can only read the certificate one time from left to right

The s-t connectivity problem

- $\mathrm{STCONN} = \{\langle G, s, t\rangle : G \text{ is a digraph, } s \text{ and } t \text{ are vertices,}$
  $\text{and there is a directed path from } s \text{ to } t\}$
- **Claim:** $\mathrm{STCONN} \in \mathrm{NL}$
- **Proof sketch:** Take a nondeterministic walk through G starting from s for $|V|$ steps. If we ever reach t, accept; otherwise, reject.
- Verifier perspective: Certificate = path from s to t

Two surprises about $\mathbf{NL}$

- We expect that $\mathrm{P} \neq \mathrm{NP}$. However, in the space complexity world…

Savitch's Theorem: $\mathrm{NL} \subseteq \mathrm{SPACE}(\log^2 n)$

- We expect that $\mathrm{NP} \neq \mathrm{coNP}$. However, in the space complexity world…

Immerman-Szelepcsényi Theorem: $\mathrm{NL} = \mathrm{coNL}$

Proof of Savitch's theorem

Savitch's Theorem: $\mathrm{NL} \subseteq \mathrm{SPACE}(\log^2 n)$

- Proof step 1: Show that $\mathrm{STCONN} \in \mathrm{SPACE}(\log^2 n)$
- Proof step 2: Show that STCONN is "NL-complete"

Savitch's algorithm

- **Claim (Savitch's algorithm)**: $\mathrm{STCONN} \in \mathrm{SPACE}(\log^2 n)$
- **Proof sketch:** Let's figure out: is there a path from s to t of length at most 2^k?
  1. For all $m \in V$:
     a) Recursively figure out whether there is a path from s to m of length at most 2^{k-1}
     b) Recursively figure out whether there is a path from m to t of length at most 2^{k-1}
     c) If both such paths exist, halt and accept
  2. Halt and reject
- Space complexity is $O(k \log n)$, which is $O(\log^2 n)$ when $k = \lceil \log |V| \rceil$

Proof of Savitch's theorem

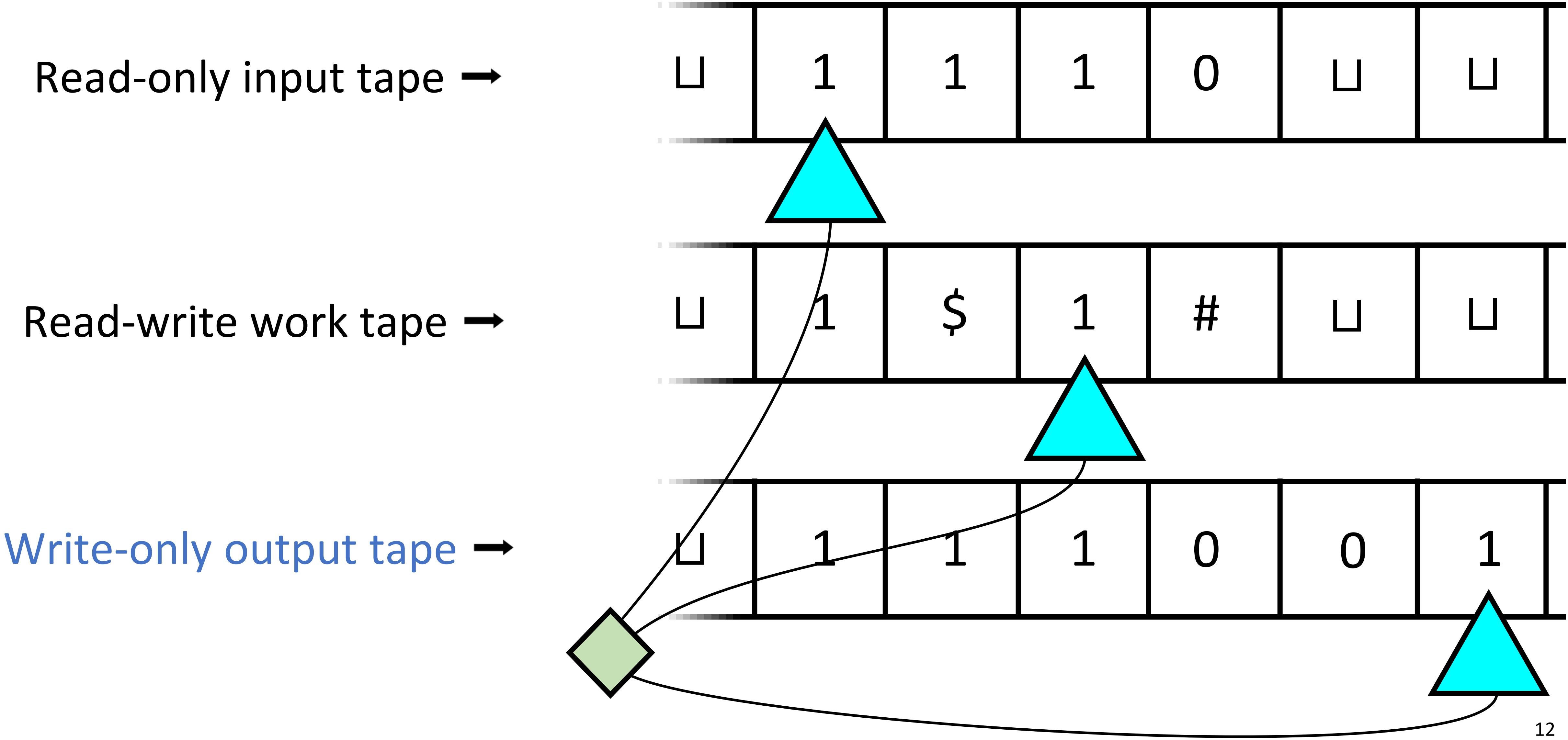
Savitch's Theorem: $\mathrm{NL} \subseteq \mathrm{SPACE}(\log^2 n)$

- Proof step 1: Show that $\mathrm{STCONN} \in \mathrm{SPACE}(\log^2 n)$ ✓
- Proof step 2: Show that STCONN is "NL-complete"

Log-space reductions

- To prove Savitch's theorem, we will use a new type of reduction
- Let $Y_1, Y_2 \subseteq \{0,1\}^*$
- **Definition:** We write $Y_1 \leq_{\mathrm{L}} Y_2$ if there exists $\Psi: \{0,1\}^* \to \{0,1\}^*$ such that
  - For every $w \in Y_1$, we have $\Psi(w) \in Y_2$ ("YES maps to YES")
  - For every $w \notin Y_1$, we have $\Psi(w) \notin Y_2$ ("NO maps to NO")
  - Ψ can be computed in $O(\log n)$ space ⬅ Definition on next slides

Space-bounded "transducer"



Space complexity for string-valued functions

- Let $\Psi: \{0,1\}^* \to \{0,1\}^*$ and let $S: \mathbb{N} \to \mathbb{N}$
- **Def:** We say Ψ is computable in $O(S)$ space if there is a 3-tape TM M such that:
  - If we initialize M with w on tape 1, then it halts with $\Psi(w)$ on tape 3
  - M never modifies tape 1 and M's behavior does not depend on what it reads on tape 3
  - The tape 1 head is always located within one cell of the input
  - When the input has length n, the tape 2 head visits $O(S(n))$ cells

NL-completeness

- Let $Y \subseteq \{0, 1\}^*$
- **Definition:** We say that Y is NL-complete if $Y \in \mathrm{NL}$ and for every $Z \in \mathrm{NL}$, we have $Z \leq_{\mathrm{L}} Y$

STCONN is NL-complete

Theorem: STCONN is NL-complete

- **Proof:** We have already shown $\text{STCONN} \in \text{NL}$
- Now let M be a nondeterministic log-space TM that decides Y
- Reduction: $\Psi(w) = \langle G, s, t \rangle$
- Each vertex in G represents a "configuration" of M on w, namely, the internal state, the contents of the work tape, and the locations of heads

STCONN is NL-complete

- We put an edge from u to v if M can go from u to v in a single step (with w written on input tape)
- We let $s =$ the initial configuration and $t =$ the accepting configuration
- (Without loss of generality, the accepting configuration is unique)
- YES maps to YES ✔️ NO maps to NO ✔️
- Exercise: The reduction can be computed in $O(\log n)$ space ✔️

Proof of Savitch's theorem

Savitch's Theorem: $\mathrm{NL} \subseteq \mathrm{SPACE}(\log^2 n)$

- Proof step 1: Show that $\mathrm{STCONN} \in \mathrm{SPACE}(\log^2 n)$ ✓
- Proof step 2: Show that STCONN is NL-complete ✓
- Proof step 3: Show that $\mathrm{SPACE}(\log^2 n)$ is closed under log-space mapping reductions

Composing space-bounded algorithms

- Let $f:\{0,1\}^* \to \{0,1\}^*$ be a function computable in space $O(S_f)$ where $S_f:\mathbb{N} \to \mathbb{N}$
- Let $g:\{0,1\}^* \to \{0,1\}^*$ be a function computable in space $O(S_g)$ where $S_g:\mathbb{N} \to \mathbb{N}$
- Assume S_g is monotone increasing and $S_g(n) \geq \log n$
- Define $\ell(n) = \max_{w \in \{0,1\}^n} |f(w)|$

Lemma: $g \circ f$ is computable in space $O\left(S_f(n) + S_g(\ell(n))\right)$

Composing space-bounded algorithms

Lemma: $g \circ f$ is computable in space $O\left(S_f(n) + S_g(\ell(n))\right)$

- Let M_f, M_g be the TMs that compute f, g. Our job is to simulate M_g on $f(w)$
- Key challenge: We cannot afford to write $f(w)$ down!
- We remember the location of M_g's input-tape head, i, in our work space
- To simulate a single step of M_g, first we need to compute $f(w)_i$
- To compute it, we simulate M_f on w and discard all but i-th output symbol!

Composing space-bounded algorithms

Lemma: $g \circ f$ is computable in space $O\left(S_f(n) + S_g(\ell(n))\right)$

- How much space did we use?
- $S_g(\ell(n))$ space used by simulated M_g
- $S_f(n)$ space used by simulated M_f
- $O(\log(\ell(n)))$ space to keep track of the location of M_g's input-tape head and M_f's output-tape head

Closure under reductions

- **Corollary:** Let $Y_1, Y_2 \subseteq \{0,1\}^*$. If $Y_1 \leq_{\mathrm{L}} Y_2$ and $Y_2 \in \mathrm{SPACE}(\log^2 n)$, then $Y_1 \in \mathrm{SPACE}(\log^2 n)$
- **Proof:** The log-space reduction Ψ runs in polynomial time (cf. $\mathrm{L} \subseteq \mathrm{P}$)
- Therefore, $|\Psi(w)| \leq \mathrm{poly}(|w|)$
- By the lemma, $Y_1 \in \mathrm{SPACE}\left(\log n + \log^2\left(\mathrm{poly}(n)\right)\right) = \mathrm{SPACE}(\log^2 n)$

Proof of Savitch's theorem

Savitch's Theorem: $\mathrm{NL} \subseteq \mathrm{SPACE}(\log^2 n)$

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- Proof step 2: Show that STCONN is NL-complete ✓
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