CMSC 28100

Introduction to Complexity Theory

Spring 2025
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Homework

- Exercises 1-4 are available in Canvas (due on Tuesday)
- If you aren’t officially enrolled, send me an email
- Office hours (Thursday, Friday, Monday) are a good place to find study partners / homework collaborators

Which problems

can be solved

through computation?

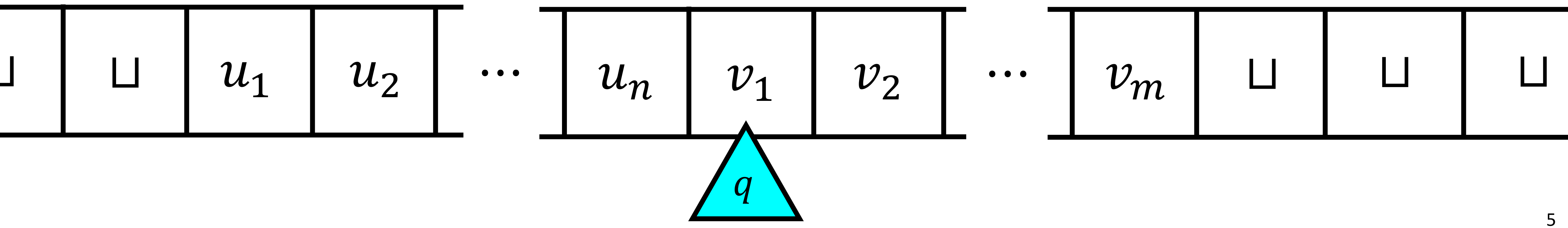
Defining Turing machines rigorously

- **Definition**: A Turing machine is a 7-tuple $M = (Q, q_0, q_{\text{accept}}, q_{\text{reject}}, \Sigma, ⊔, \delta)$ such that
  - Q is a finite set (the set of "states")
  - $q_0, q_{\text{accept}}, q_{\text{reject}} \in Q$ and $q_{\text{accept}} \neq q_{\text{reject}}$
  - Σ is a finite set of symbols (the "tape alphabet")
  - $⊔$ is a symbol (the "blank symbol")
  - $\{0, 1, ⊔\} \subseteq \Sigma$ and $⊔ \notin \{0, 1\}$
  - δ is a function $\delta: Q \times \Sigma \to Q \times \Sigma \times \{\mathrm{L}, \mathrm{R}\}$ (the "transition function")

⚠️ Warning: The definition in the textbook is slightly different. Sorry! (The two models are equivalent.)

Configurations of a Turing machine

- Let $M = (Q, q_0, q_{\text{accept}}, q_{\text{reject}}, \Sigma, \sqcup, \delta)$ be a Turing machine
- A configuration is a triple (u, q, v) where $u \in \Sigma^*$, $q \in Q$, $v \in \Sigma^*$, and $v \neq \epsilon$
- Interpretation:
  - The tape currently contains $\cdots \sqcup\sqcup\sqcup\sqcup\ uv\ \sqcup\sqcup\sqcup\sqcup \cdots$
  - The machine is currently in state q and the head is pointing at the first symbol of v



The initial configuration

- Let $w \in \{0, 1\}^*$ be an input
- The initial configuration of M on w is

$$\begin{cases} q_0 w & \text{if } w \neq \epsilon \\ q_0 \sqcup & \text{if } w = \epsilon \end{cases}$$

The "next" configuration

- For any configuration uqv, we define $\mathrm{NEXT}(uqv)$ as follows:
  - Break uqv into individual symbols: $uqv = u_1 u_2 \dots u_{n-1} u_n q v_1 v_2 v_3 \dots v_m$
  - If $\delta(q, v_1) = (q', b, \mathrm{R})$, then $\mathrm{NEXT}(uqv) = u_1 u_2 \dots u_{n-1} u_n b q' v_2 v_3 \dots v_m$
    - Edge case: If $m = 1$, then $\mathrm{NEXT}(uqv) = u_1 u_2 \dots u_{n-1} u_n b q' \sqcup$
  - If $\delta(q, v_1) = (q', b, \mathrm{L})$, then $\mathrm{NEXT}(uqv) = u_1 u_2 \dots u_{n-1} q' u_n b v_2 v_3 \dots v_m$
    - Edge case: If $n = 0$, then $\mathrm{NEXT}(uqv) = q' \sqcup b' v_2 v_3 \dots v_m$
- We write $\mathrm{NEXT}_M(uqv)$ if M is not clear from context

Halting configurations

- An accepting configuration is a configuration of the form $uq_{\text{accept}}v$
- A rejecting configuration is a configuration of the form $uq_{\text{reject}}v$
- A halting configuration is an accepting or rejecting configuration

Computation history

- Let $w \in \{0,1\}^*$ be an input
- Let C_0 be the initial configuration of M on w
- Inductively, for each $i \in \mathbb{N}$, let $C_{i+1} = \mathrm{NEXT}(C_i)$
- The computation history of M on w is the sequence $C_0, C_1, \ldots, C_T$, where C_T is the first halting configuration in the sequence
- If there is no such C_T, then the computation history is $C_0, C_1, C_2, \ldots$ (infinite)

Accepting, rejecting, and looping

- If the computation history of M on w ends with an accepting configuration, then we say that M accepts w
- If the computation history of M on w ends with a rejecting configuration, then we say that M rejects w
- In either of those cases, we say that M halts on w. If the computation history of M on w is infinite, then we say that M loops on w

Time

- Suppose the computation history of M on w is $C_0, C_1, \dots, C_T$
- We say that T is the running time of M on w
- If M loops on w, then its running time on w is ∞
- We say that M halts on w within T steps if the running time of M on w is at most T

Space

- The space used by M on w is the number of cells that are "used"
  - I.e., the head visits the cell OR the cell initially contains an input bit
  - (Can be ∞)
- Formally, let $C_0, C_1, \ldots$ b … on w
- Write $C_i = (u_i, q_i, v_i)$ w …
- The space used by M on w is $\max_i |u_i v_i|$

Which of the following statements is <u>false</u>?

A: Space used on w is at most $|w| + 1 +$ running time on w

B: If M halts on w within $|w|$ steps, then M halts on ww

C: If M halts on w, then M uses a finite amount of space on w

D: If M uses a finite amount of space on w, then M halts on w

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Which problems

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Languages

- A binary language is a set $Y \subseteq \{0,1\}^*$
- Each language $Y \subseteq \{0,1\}^*$ represents a distinct computational problem:

  “Given $w \in \{0,1\}^*$, figure out whether $w \in Y$”

Deciding a language

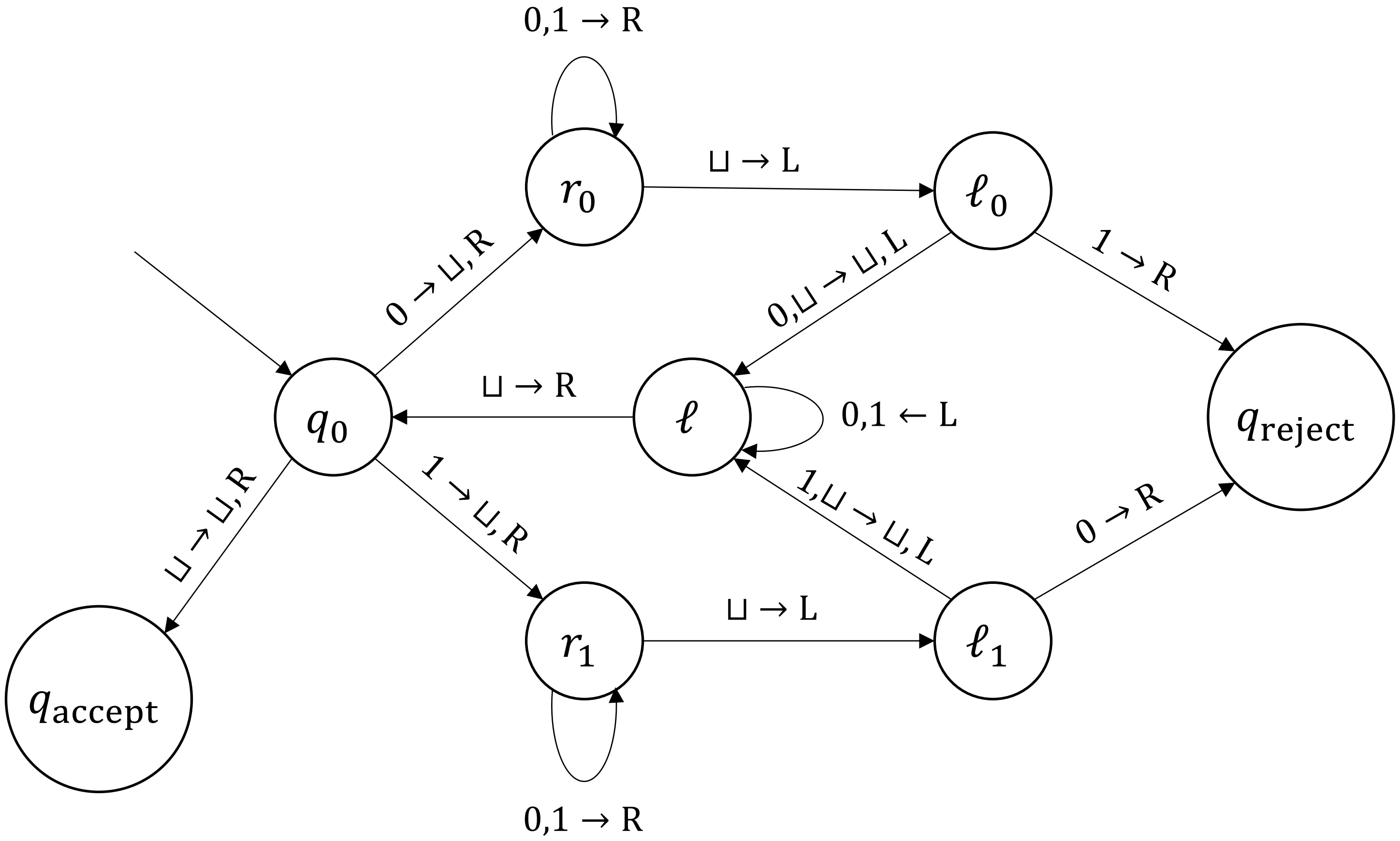
- Let M be a Turing machine and let $Y \subseteq \{0,1\}^*$
- We say that M decides Y if
  - M accepts every $w \in Y$, and
  - M rejects every $w \in \{0,1\}^* \setminus Y$
- This is a mathematical model of what it means to "solve a problem"

Example: Palindromes

- **Informal problem statement:** "Given $w \in \{0,1\}^*$, determine whether w is the same forward and backward."
- **The same problem, formulated as a language:**

$$\mathrm{PALINDROMES} = \{w \in \{0,1\}^* : w \text{ is the same forward and backward}\}$$

Example: A TM that decides PALINDROMES



Example: Parity

- **Informal problem statement:** "Given $w \in \{0,1\}^*$, determine whether the number of ones in w is even or odd."
- **The same problem, formulated as a language:**

$$\mathrm{PARITY} = \{w \in \{0,1\}^* : w \text{ has an odd number of ones}\}$$

Example: A TM that decides PARITY

- Let $M = (Q, q_0, h_1, h_0, \Sigma, \sqcup, \delta)$, where $Q = \{q_0, q_1, h_0, h_1\}$, $\Sigma = \{0, 1, \sqcup\}$, and

$$\delta(q_a, b) = \begin{cases} (q_{a+b}, b, \mathrm{R}) & \text{if } b \in \{0, 1\} \\ (h_a, b, \mathrm{R}) & \text{if } b = \sqcup \end{cases}$$

(Addition is mod 2)

- **Claim:** M decides $\mathrm{PARITY} \coloneqq \{w \in \{0,1\}^* : w \text{ has an odd number of ones}\}$

- **Proof sketch:** Let $C_0, C_1, \ldots$ be the computation history of M on $w \in \{0,1\}^n$
- By induction on i, we have $C_i = w_1 w_2 \ldots w_i q_{w_1+\cdots+w_i} w_{i+1} \ldots w_n$ for all $i < n$
- Consequently, $C_n = w q_{w_1+\cdots+w_i} \sqcup$ and $C_{n+1} = w \sqcup h_{w_1+\cdots+w_n} \sqcup$

Example: Primality testing

- **Informal problem statement:** "Given $K \in \mathbb{N}$, determine whether K is prime."
- **Formulating the problem as a language:**
  - Let $\langle K \rangle$ denote the binary encoding of K, i.e., the standard base-2 representation of K
  - Example: $\langle 6 \rangle = 110$. Note that $K \in \mathbb{N}$ whereas $\langle K \rangle \in \{0,1\}^*$
  - Language:

$$\mathrm{PRIMES} = \{\langle K \rangle : K \text{ is a prime number}\}$$

Encoding the input as a string

"This is not a pipe."
(1929 painting by René Magritte)

- **OBJECTION:** "The fact that I have to encode the input before feeding it into a Turing machine seems fishy."
- **RESPONSE:** The same is true of human computation!
- We say, "Given a natural number, determine whether it is prime," but is it truly possible to "give" someone an abstract concept such as a number?
- Being pedantic, we could speak more precisely and say, "Given a piece of text, determine whether it represents/encodes a prime number"

Larger alphabets

- **OBJECTION:** "Fine, the input needs to be encoded as a string. But why does it have to be a binary string? What about larger alphabets?"
- **RESPONSE 1:** The Turing machine definition can be modified to handle inputs over other alphabets**.** We focus on binary inputs for simplicity's sake
- **RESPONSE 2:** We can always encode symbols from other alphabets in binary

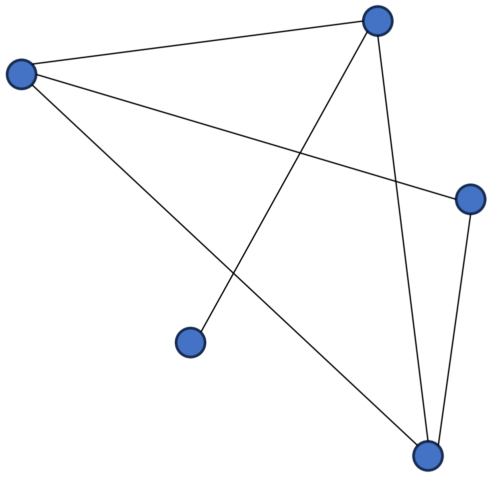
Example: ASCII

[NUL]	[SOH]	[STX]	[ETX]	[EOT]	[ENQ]	[ACK]	[BEL]	[BS]	[HT]	[LF]	[VT]	[FF]
0000000	0000001	0000010	0000011	0000100	0000101	0000110	0000111	0001000	0001001	0001010	0001011	0001100
[CR]	[SO]	[SI]	[DLE]	[DC1]	[DC2]	[DC3]	[DC4]	[NAK]	[SYN]	[ETB]	[CAN]	[EM]
0001101	0001110	0001111	0010000	0010001	0010010	0010011	0010100	0010101	0010110	0010111	0011000	0011001
[SS]	[ESC]	[FS]	[GS]	[RS]	[US]	[SPACE]	!	"	#	$	%	&
0011010	0011011	0011100	0011101	0011110	0011111	0100000	0100001	0100010	0100011	0100100	0100101	0100110
'	(	)	*	+	,	-	.	/	0	1	2	3
0100111	0101000	0101001	0101010	0101011	0101100	0101101	0101110	0101111	0110000	0110001	0110010	0110011
4	5	6	7	8	9	:	;	<	=	>	?	@
0110100	0110101	0110110	0110111	0111000	0111001	0111010	0111011	0111100	0111101	0111110	0111111	1000000
A	B	C	D	E	F	G	H	I	J	K	L	M
1000001	1000010	1000011	1000100	1000101	1000110	1000111	1001000	1001001	1001010	1001011	1001100	1001101
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1001110	1001111	1010000	1010001	1010010	1010011	1010100	1010101	1010110	1010111	1011000	1011001	1011010
[	\	]	^	_	`	a	b	c	d	e	f	g
1011011	1011100	1011101	1011110	1011111	1100000	1100001	1100010	1100011	1100100	1100101	1100110	1100111
h	i	j	k	l	m	n	o	p	q	r	s	t
1101000	1101001	1101010	1101011	1101100	1101101	1101110	1101111	1110000	1110001	1110010	1110011	1110100
u	v	w	x	y	z	{	|	}	~	[DEL]		
1110101	1110110	1110111	1111000	1111001	1111010	1111011	1111100	1111101	1111110	1111111		

Another encoding example: Connectivity

- **Informal problem statement:** "Given a K-vertex graph G, determine whether it is connected"
- **Formulating the problem as a language:**
  - Let $\langle G \rangle \in \{0,1\}^{K^2}$ denote the adjacency matrix of G
  - Language:

$$\mathrm{CONNECTED} = \{\langle G \rangle : G \text{ is a connected graph}\}$$



Multiple possible encodings

- **OBJECTION:** “Why are we using adjacency matrices instead of adjacency lists?”
- **RESPONSE:** It doesn’t matter much which encoding we use, because it is not hard to convert between the two encodings

Encoding other things as strings

- General convention: If X is any mathematical object that can be written down (a number, a graph, a polynomial, …), then we use the notation $\langle X \rangle$ to denote some "reasonable" encoding of X as a binary string
- It typically doesn't matter which specific encoding we use, provided we choose something reasonable
- If you are unsure how $\langle X \rangle$ should be defined in a particular case, ask!

Invalid inputs

- **Informal problem statement:** "Given a graph G, determine whether it is connected"
- **The same problem, formulated as a language:**

$$\text{CONNECTED} = \{\langle G \rangle : G \text{ is a connected graph}\}$$

What if we are given $w \in \{0,1\}^*$ that is not the encoding of any graph?

A: This situation cannot occur

B: It doesn't matter what we do

C: We can accept or reject, but we must not loop

D: We must reject

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ph, we should accept

graph, we should reject