CMSC 28100

Introduction to Complexity Theory

Autumn 2025
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Midterm exam

- Midterm exam will be in class on **Friday, October 24**
- To prepare for the midterm, you only need to study the material prior to this point
- The midterm will be about Turing machines, decidability, and undecidability

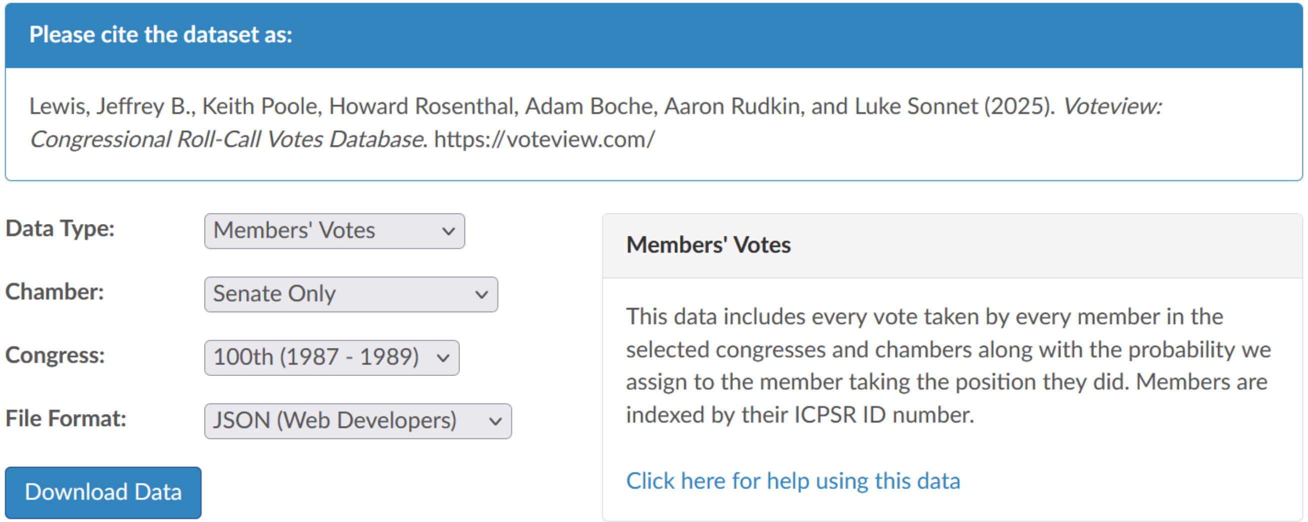
Which problems

can be solved

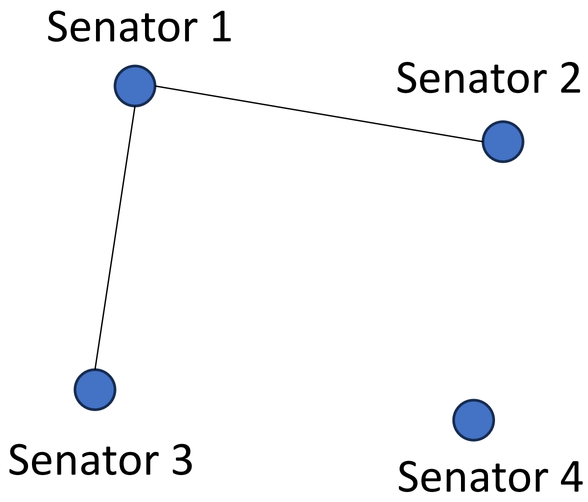
through computation?

Applying our theory

- **Question:** In the year 1988, were there 50 U.S. senators, every pair of which voted the same way more than 50% of the time?
- **Step 1:** Gather data



Agreement graph



- **Step 2:** Construct "agreement graph"
- Edge $\{u, v\}$ means that senators u and v agreed on most votes
- **Question:** Are there 50 vertices in this graph that are all adjacent to one another?

The clique problem

- A k-clique in a graph $G = (V, E)$ is a set $S \subseteq V$ such that $|S| = k$ and every two vertices in S are connected by an edge

- Example: This graph has a 4-clique

Which of the following statements is <u>false</u>?

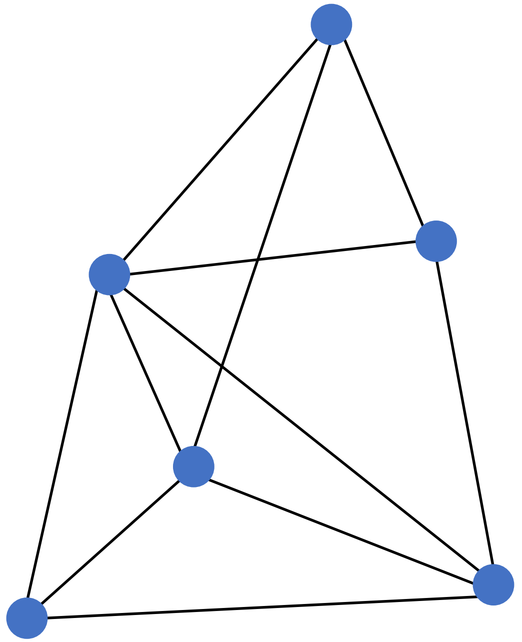
A: Every vertex in a k-clique has degree at least $k-1$

B: A single graph might have many k-cliques

C: If G has fewer than $\binom{k}{2}$ edges, then G does not have a k-clique

D: If every vertex has degree at least $k-1$, then G has a k-clique

Respond at PollEv.com/whoza or text “whoza” to 22333



The clique problem

- Let $\text{CLIQUE} = \{\langle G, k\rangle : G \text{ has a } k\text{-clique}\}$
- Example: Let G be the graph with the following adjacency matrix
- Does G have a 4-clique?

	a	**b**	**c**	**d**	**e**	**f**	**g**
a	0	1	1	0	0	1	0
b	1	0	0	1	1	0	1
c	1	0	0	0	1	0	1
d	0	1	0	0	1	0	1
e	0	1	1	1	0	1	1
f	1	0	0	0	1	0	1
g	0	1	1	1	1	1	0

The clique problem

- Let $\text{CLIQUE} = \{\langle G, k\rangle : G \text{ has a } k\text{-clique}\}$
- Example: Let G be the graph with the following adjacency matrix
- Does G have a 4-clique?

Is CLIQUE decidable?

A: Yes

B: No

C: It depends on whether $\langle G \rangle$ is an adjacency matrix or adjacency list

D: It's not a language, so the question doesn't make sense

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	a	b	c	d	e	f	g
a	0	1	1	0	0	1	0
b	1	0	0	1	1	0	1
c	1	0	0	0	1	0	1
d	0	1	0	0	1	0	1
e	0	1	1	1	0	1	1
f	1	0	0	0	1	0	1
g	0	1	1	1	1	1	0

The clique problem

- Let $\text{CLIQUE} = \{\langle G, k\rangle : G \text{ has a } k\text{-clique}\}$
- **Claim:** CLIQUE is decidable
- **Proof sketch:** Given $\langle G, k\rangle$ where $G = (V, E)$, try all possible subsets $S \subseteq V$
  - Check whether $|S| = k$
  - Check whether $\{u, v\} \in E$ for every $u, v \in S$ such that $u \neq v$
- If we find a k-clique, accept; otherwise, reject.

The clique problem

- **Question:** In the year 1988, were there 50 U.S. senators, every pair of which voted the same way more than 50% of the time?
- **Step 1:** Gather data ✔
- **Step 2:** Construct agreement graph ✔
- **Step 3:** Apply CLIQUE algorithm

Our algorithm is so slow that it’s worthless

- **Question:** In the year 1988, were there 50 U.S. senators, every pair of which voted the same way more than 50% of the time?
- Checking all possible sets of senators would take longer than a lifetime!
- One begins to feel that CLIQUE might as well be undecidable!

Which problems

can be solved

through computation?

Refining our model

- Our model so far: Decidable vs. undecidable
- Now we will refine our model
  - We only have a limited amount of time (and other resources)
- “Complexity theory” vs. “Computability theory”

Time complexity

- Let M be a Turing machine
- The time complexity of M is a function $T_M \colon \mathbb{N} \to \mathbb{N}$

$$T_M(n) \coloneqq \max_{w \in \{0,1\}^n} (\text{running time of } M \text{ on } w)$$

- We focus on the worst-case n-bit input

Scaling behavior

- We focus on the limiting behavior of $T_M(n)$ as $n \to \infty$
- How "quickly" does the running time increase when we consider larger and larger inputs?

Asymptotic analysis

- Two possible time complexities:

$$T_1(n) = 3n^2 + 14$$

$$T_2(n) = 2n^2 + 64n + \lceil \sqrt{n} \rceil$$

- When n is large, the leading $C \cdot n^2$ term dominates
- We will ignore the low-order terms and the leading coefficient C
- We focus on the n^2 part ("quadratic time")

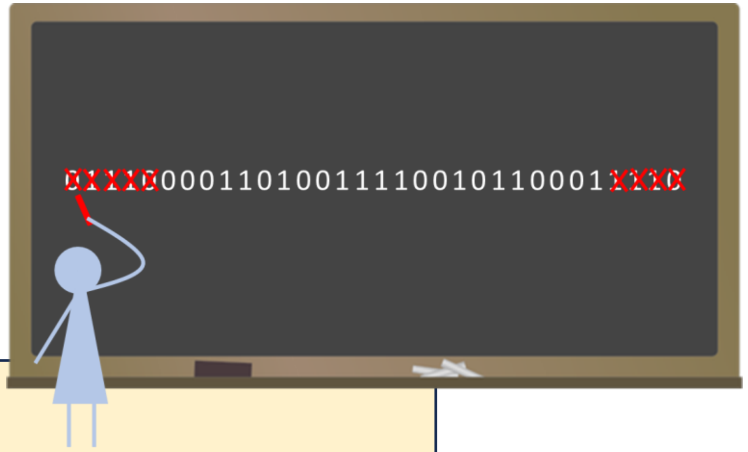
Big-O notation

- Let $T, f: \mathbb{N} \to [0, \infty)$ be any two functions
- **Definition:** We say that T is $O(f)$ if there exist $C, n_* \in \mathbb{N}$ such that for every $n > n_*$, we have $T(n) \leq C \cdot f(n)$
- Notation: $T \in O(f)$ or $T \leq O(f)$ or $T = O(f)$

Big-O notation examples

- $3n^2 + 14$ is $O(n^2)$
- $3n^2 + 14$ is $O(n^2 + n)$
- $3n^2 + 14$ is $O(n^3)$
- $3n^2 + 14$ is not $O(n^{1.9})$

Example: Palindromes



Claim: There exists a Turing machine that decides PALINDROMES with time complexity $O(n^2)$.

- **Proof sketch:** Recall our Turing machine that decides PALINDROMES
- At most $n/2$ back-and-forth passes over the input
- Each back-and-forth pass takes $O(n)$ steps
- Total time complexity: $O(n) \cdot n/2 = O(n^2)$

Optimality

- Is there a faster Turing machine that decides $\mathrm{PALINDROMES}$?
- Answer: No
- Use big-Ω notation to make this precise

Big-Ω

- Let $T, f: \mathbb{N} \to [0, \infty)$ be any two functions
- We say that T is $\Omega(f)$ if there exist $c \in (0,1)$ and $n_* \in \mathbb{N}$ such that for every $n > n_*$, we have $T(n) \geq c \cdot f(n)$
- Example: $0.1n^2 + 14$ is $\Omega(n^2)$ and $\Omega(n)$, but not $\Omega(n^3)$

Palindromes time complexity lower bound

- Let M be a one-tape Turing machine

Theorem: If M decides $\mathrm{PALINDROMES}$, then the time complexity of M is $\Omega(n^2)$.

- (Proof omitted)

Palindromes, revisited

Claim: There exists a two-tape Turing machine M that decides $\mathrm{PALINDROMES}$ with time complexity $O(n)$.

- **Proof sketch:**
  1. Copy the input to tape 2
  2. Scan tape 1 from left to right and scan tape 2 from right to left to compare

Multi-tape Turing machines, revisited

- Multi-tape TMs can decide $\mathrm{PALINDROMES}$ in $O(n)$ time…
- But single-tape TMs require $\Omega(n^2)$ time!
- So multi-tape / single-tape TMs are not equivalent after all?

Exponential vs. polynomial

- In this course, we are not concerned with the distinction between $O(n)$ time and $O(n^2)$ time
  - We're happy with either
- Our focus: The distinction between a polynomial time complexity, such as $T(n) = n^2$, and an exponential time complexity, such as $T(n) = 2^n$
  - Usable vs. useless

Exponential vs. polynomial

- We write $T(n) = \text{poly}(n)$ if there is some k such that $T(n) = O(n^k)$
- Exponentials grow much faster than polynomials!
- We can make this precise using little-o and little-ω notation

Little-o notation

- Let $T, f: \mathbb{N} \to [0, \infty)$ be any two functions
- We say that T is $o(f)$ if for every $c \in (0, 1)$, there exists $n_* \in \mathbb{N}$ such that for every $n > n_*$, we have $T(n) < c \cdot f(n)$
- Equivalent:

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} = 0$$

Little-ω notation

- Let $T, f: \mathbb{N} \to [0, \infty)$ be any two functions
- We say that T is $\omega(f)$ if for every $C \in \mathbb{N}$, there exists $n_* \in \mathbb{N}$ such that for every $n > n_*$, we have $T(n) > C \cdot f(n)$
- Equivalent:

$$\lim_{n\to\infty} \frac{T(n)}{f(n)} = \infty$$

Exponential vs. polynomial

Claim: For every constant $k \in \mathbb{N}$, we have $n^k = o(2^n)$

- **Proof:** If $n \geq k+1$, then

$$2^n = \#\text{ subsets of } \{1, 2, \dots, n\} = \sum_{i=0}^{n} \binom{n}{i} \geq \binom{n}{k+1} \geq \left(\frac{n}{k+1}\right)^{k+1}$$

$$= \Omega\left(n^{k+1}\right)$$

$$= \omega\left(n^k\right).$$