CMSC 28100

Introduction to Complexity Theory

Autumn 2025
Instructor: William Hoza

Modified Post's Correspondence Problem

- **Given:** A list of "dominos" $\boxed{\boxed{\begin{matrix} t_1 \\ b_1 \end{matrix}}}\, , \boxed{\begin{matrix} t_2 \\ b_2 \end{matrix}}\, , \boxed{\begin{matrix} t_3 \\ b_3 \end{matrix}}\, , \cdots , \boxed{\begin{matrix} t_k \\ b_k \end{matrix}}$
- **Goal:** Determine whether it is possible to construct a "match"
- A "match" is a sequence of dominos $\boxed{\boxed{\begin{matrix} t_1 \\ b_1 \end{matrix}}} \boxed{\begin{matrix} t_{i_1} \\ b_{i_1} \end{matrix}} \boxed{\begin{matrix} t_{i_2} \\ b_{i_2} \end{matrix}} \boxed{\begin{matrix} t_{i_3} \\ b_{i_3} \end{matrix}} \boxed{\begin{matrix} t_{i_4} \\ b_{i_4} \end{matrix}} \boxed{\begin{matrix} t_{i_5} \\ b_{i_5} \end{matrix}} \cdots \boxed{\begin{matrix} t_{i_n} \\ b_{i_n} \end{matrix}}$

such that $t_1 t_{i_1} t_{i_2} \cdots t_{i_n} = b_1 b_{i_1} b_{i_2} \cdots b_{i_n}$

- Using the same domino multiple times is permitted

Lemma: MPCP is undecidable

Proof that MPCP is undecidable

- Assume there is a TM P that decides MPCP
- Let's construct a new TM H that decides HALT

H

Given $\langle M, w \rangle$:

1. Construct dominos $t_1, \dots, t_k, b_1, \dots b_k$ based on M and w (details on next slide)
2. Simulate P on $\langle t_1, \dots, t_k, b_1, \dots, b_k \rangle$
3. If P accepts, accept. If P rejects, reject.

The dominos for $\langle M, w \rangle$

- $\left[\dfrac{\epsilon}{(q_0 \sqcup w)}\right]$, $\left[\dfrac{(}{(}\right]$, $\left[\dfrac{)}{)}\right]$, $\left[\dfrac{(q_{\text{accept}})}{\epsilon}\right]$, and $\left[\dfrac{(q_{\text{reject}})}{\epsilon}\right]$
- For every $q \in Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}$ and every $b \in \Sigma$:
  - If $\delta(q, b) = (q', b', \mathrm{R})$, we include $\left[\dfrac{qb)}{b'q' \sqcup)}\right]$, and we include $\left[\dfrac{qba}{b'q'a}\right]$ for every $a \in \Sigma$
  - If $\delta(q, b) = (q', b', \mathrm{L})$, we include $\left[\dfrac{(qb}{(q' \sqcup b'}\right]$, and we include $\left[\dfrac{aqb}{q'ab'}\right]$ for every $a \in \Sigma$
- $\left[\dfrac{b}{b}\right]$, $\left[\dfrac{bq_{\text{accept}}}{q_{\text{accept}}}\right]$, $\left[\dfrac{q_{\text{accept}}b}{q_{\text{accept}}}\right]$, $\left[\dfrac{q_{\text{reject}}b}{q_{\text{reject}}}\right]$, and $\left[\dfrac{bq_{\text{reject}}}{q_{\text{reject}}}\right]$ for every $b \in \Sigma$

Proof that MPCP is undecidable

- Assume there is a TM P that decides MPCP
- Let's construct a new TM H that decides HALT

H

Given $\langle M, w\rangle$:

1. Construct dominos $t_1, \ldots, t_k, b_1, \ldots b_k$ based on M and w (details on preceding slides)
2. Simulate P on $\langle t_1, \ldots, t_k, b_1, \ldots, b_k\rangle$
3. If P accepts, accept. If P rejects, reject.

Need to show:

- If M halts on w, then there is a match
- If there is a match, then M halts on w

Domino Feature 1

- **Domino Feature 1:** For every non-halting configuration C of M, there is a sequence of dominos such that the top string is (C) and bottom string is $(\mathrm{NEXT}(C))$
- Proof omitted, but here's an example:

(	0	1	#	0	$0q_10$	0	⊔	)
(	0	1	#	0	q_201	1	⊔	)

- Think of this sequence as one "super-domino"

(C)
$(\mathrm{NEXT}(C))$

If M halts on w, then there is a match

- Let $C_0, \dots, C_T$ be the halting computation history of M on w
- Match:

ϵ	(C_0)	(C_1)	…	(C_{T-1})	(C_T)	(D_1)	(D_2)	…	(D_{n-1})	(D_n)
(C_0)	(C_1)	(C_2)		(C_T)	(D_1)	(D_2)	(D_3)		(D_n)	ϵ

- $|D_i + 1| = |D_i| - 1$

Proof that MPCP is undecidable

- Assume there is a TM P that decides MPCP
- Let's construct a new TM H that decides HALT

H

Given $\langle M, w\rangle$:

1. Construct dominos $t_1, \ldots, t_k, b_1, \ldots b_k$ based on M and w (details on preceding slides)
2. Simulate P on $\langle t_1, \ldots, t_k, b_1, \ldots, b_k\rangle$
3. If P accepts, accept. If P rejects, reject.

Need to show:

- If M halts on w, then there is a match
- If there is a match, then M halts on w

Domino Feature 3

- **Domino Feature 3:** If C is a non-halting configuration, then every sequence of dominos in which the top string starts with (C) must begin with the following super-domino: $\boxed{\begin{matrix}(C)\\ \big(\mathrm{NEXT}(C)\big)\end{matrix}}$
- Proof omitted

If there is a match, then M halts on w

- Assume there is a match
- By Domino Feature 3, it must have the form

ϵ	(C_0)	(C_1)	…	(C_{T-1})	$(C_T)x$
(C_0)	(C_1)	(C_2)		(C_T)	x

where C_T is a halting configuration and $x \in \Gamma^*$

Proof that MPCP is undecidable

- Assume there is a TM P that decides MPCP
- Let's construct a new TM H that decides HALT

H

Given $\langle M, w \rangle$:

1. Construct dominos $t_1, \dots, t_k, b_1, \dots b_k$ based on M and w (details on preceding slides)
2. Simulate P on $\langle t_1, \dots, t_k, b_1, \dots, b_k \rangle$
3. If P accepts, accept. If P rejects, reject.

Need to show:

- If M halts on w, then there is a match
- If there is a match, then M halts on w

Post's Correspondence Problem is undecidable

- **Post's correspondence problem, formulated as a language:**

$$\mathrm{PCP} = \{\langle t_1, \dots, t_k, b_1, \dots, b_k \rangle : \exists i_1, \dots, i_n \text{ such that } t_{i_1} \cdots t_{i_n} = b_{i_1} \cdots b_{i_n}\}$$

Theorem: PCP is undecidable

- Proof outline:
  - Step 1: Reduce HALT to a modified version ("MPCP") ✔
  - Step 2: Reduce MPCP to PCP

Proof that PCP is undecidable

- Assume there is a TM P that decides PCP
- Let's construct a new TM M that decides MPCP
- For a string $u = u_1 u_2 \dots u_n$, define $\overline{u} = u_1 \star u_2 \star \cdots \star u_n$

M:

Given $\boxed{\boxed{\begin{matrix} t_1 \\ b_1 \end{matrix}}}$ $\boxed{\begin{matrix} t_2 \\ b_2 \end{matrix}}$ $\boxed{\begin{matrix} t_3 \\ b_3 \end{matrix}}$ … $\boxed{\begin{matrix} t_k \\ b_k \end{matrix}}$:

1. Simulate P on $\boxed{\begin{matrix} \star\,\overline{t_1} \\ \star\,\overline{b_1}\,\star \end{matrix}}$ $\boxed{\begin{matrix} \star\,\overline{t_1} \\ \overline{b_1}\,\star \end{matrix}}$ $\boxed{\begin{matrix} \star\,\overline{t_2} \\ \overline{b_2}\,\star \end{matrix}}$ $\boxed{\begin{matrix} \star\,\overline{t_3} \\ \overline{b_3}\,\star \end{matrix}}$ … $\boxed{\begin{matrix} \star\,\overline{t_k} \\ \overline{b_k}\,\star \end{matrix}}$ $\boxed{\begin{matrix} \star \\ \epsilon \end{matrix}}$
2. If P accepts, accept. If P rejects, reject.

Given $\boxed{\left[\frac{t_1}{b_1}\right]}\ \left[\frac{t_2}{b_2}\right]\ \left[\frac{t_3}{b_3}\right]\ \cdots\ \left[\frac{t_k}{b_k}\right]$:

1. Simulate P on $\left[\frac{\star\,\overline{t_1}}{\star\,\overline{b_1}\,\star}\right]\ \left[\frac{\star\,\overline{t_1}}{\overline{b_1}\,\star}\right]\ \left[\frac{\star\,\overline{t_2}}{\overline{b_2}\,\star}\right]\ \left[\frac{\star\,\overline{t_3}}{\overline{b_3}\,\star}\right]\ \cdots\ \left[\frac{\star\,\overline{t_k}}{\overline{b_k}\,\star}\right]\ \left[\frac{\star}{\epsilon}\right]$
2. If P accepts, accept. If P rejects, reject.

- Suppose the MPCP instance has a match: $\boxed{\left[\frac{t_1}{b_1}\right]}\left[\frac{t_{i_1}}{b_{i_1}}\right]\left[\frac{t_{i_2}}{b_{i_2}}\right]\left[\frac{t_{i_3}}{b_{i_3}}\right]\left[\frac{t_{i_4}}{b_{i_4}}\right]\ \cdots\ \left[\frac{t_{i_n}}{b_{i_n}}\right]$
- Then the PCP instance also has a match: $\left[\frac{\star\,\overline{t_1}}{\star\,\overline{b_1}\,\star}\right]\left[\frac{\star\,\overline{t_{i_1}}}{\overline{b_{i_1}}\,\star}\right]\left[\frac{\star\,\overline{t_{i_2}}}{\overline{b_{i_2}}\,\star}\right]\ \cdots\ \left[\frac{\star\,\overline{t_{i_n}}}{\overline{b_{i_n}}\,\star}\right]\left[\frac{\star}{\epsilon}\right]$

Given $\boxed{\left[\frac{t_1}{b_1}\right]}$ $\left[\frac{t_2}{b_2}\right]$ $\left[\frac{t_3}{b_3}\right]$ $\cdots$ $\left[\frac{t_k}{b_k}\right]$:

1. Simulate P on $\left[\frac{\star\,\overline{t_1}}{\star\,\overline{b_1}\,\star}\right]$ $\left[\frac{\star\,\overline{t_1}}{\overline{b_1}\,\star}\right]$ $\left[\frac{\star\,\overline{t_2}}{\overline{b_2}\,\star}\right]$ $\left[\frac{\star\,\overline{t_3}}{\overline{b_3}\,\star}\right]$ $\cdots$ $\left[\frac{\star\,\overline{t_k}}{\overline{b_k}\,\star}\right]$ $\left[\frac{\star}{\epsilon}\right]$
2. If P accepts, accept. If P rejects, reject.

- Conversely, suppose the PCP instance has a match
- Must start with $\left[\frac{\star\,\overline{t_1}}{\star\,\overline{b_1}\,\star}\right]$, because that's the only domino in which the top string and bottom string start with the same symbol
- Delete all $\star$ symbols $\Rightarrow$ MPCP match

Post’s Correspondence Problem is undecidable

- **Post’s correspondence problem, formulated as a language:**

$$\mathrm{PCP} = \{\langle t_1, \ldots, t_k, b_1, \ldots, b_k\rangle : \exists i_1, \ldots, i_n \text{ such that } t_{i_1} \cdots t_{i_n} = b_{i_1} \cdots b_{i_n}\}$$

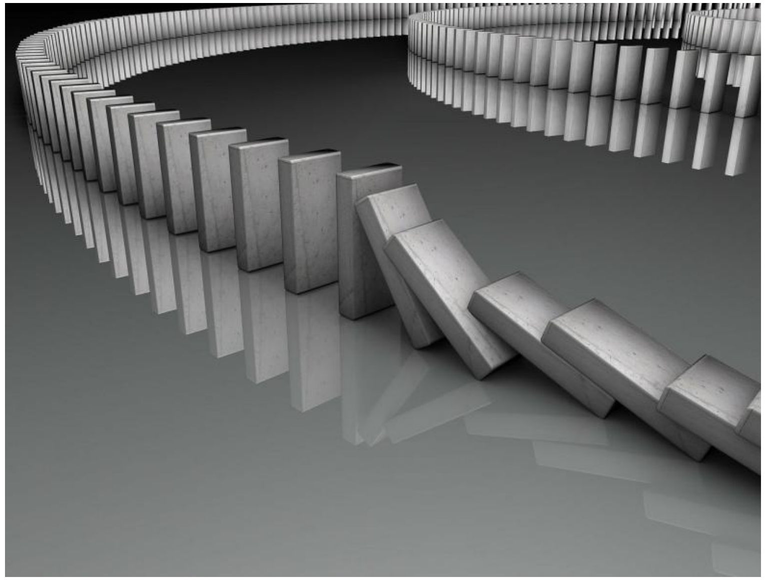
Theorem: PCP is undecidable

- Proof outline:
    - Step 1: Reduce REJECT to a modified version (“MPCP”) ✔
    - Step 2: Reduce MPCP to PCP ✔

Post’s Correspondence Problem: Recap

- Post’s Correspondence Problem seems like “just a domino puzzle”
- However, we showed how to build a computer out of dominos!
- PCP was secretly a problem about Turing machines all along!

Undecidability



- Known undecidable languages:
  - SELF-REJECTORS
  - HALT
  - PCP and MPCP
- Next: One more example

The acceptance problem

- **Informal problem statement:** Given a Turing machine M and an input w, determine whether M accepts w.
- **The same problem, formulated as a language:**

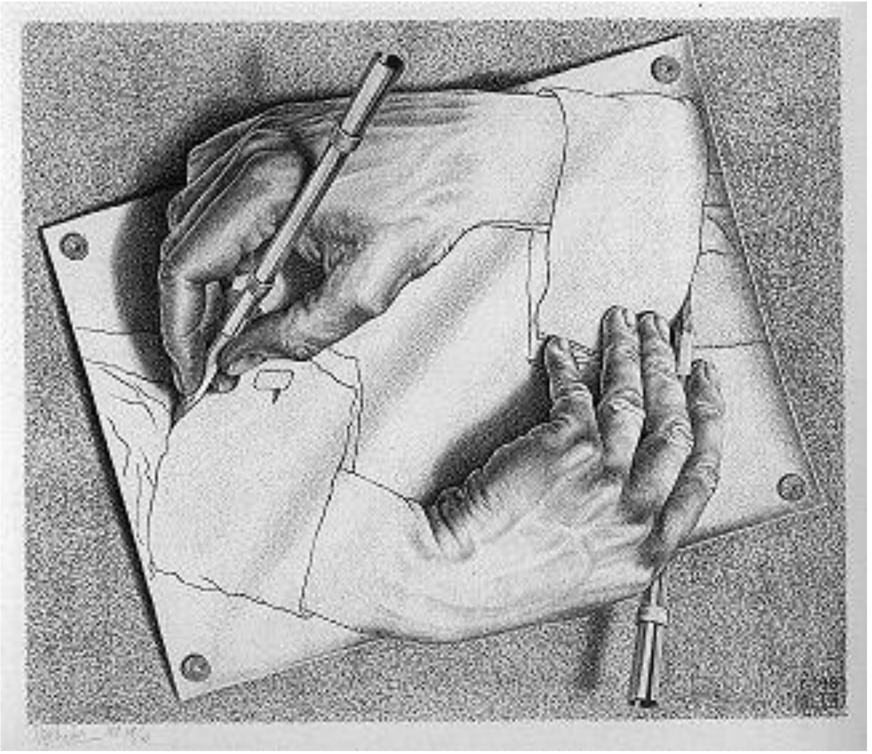
$$\text{ACCEPT} = \{\langle M, w\rangle : M \text{ is a Turing machine that accepts } w\}$$

- HALT and ACCEPT are both about predicting TMs' behavior

Theorem: ACCEPT is undecidable

Code as data II

- Our proof that ACCEPT is undecidable will involve Turing machines constructing Turing machines
- Turing machines can both read and write descriptions $\langle M \rangle$ where M is a Turing machine



"Drawing Hands."
(1948 lithograph by M. C. Escher)

Proof that ACCEPT is undecidable

- Assume there is a TM A that decides ACCEPT
- Let's construct a new TM H that decides HALT

H

Given $\langle M, w \rangle$:

1. Construct $\langle M' \rangle$, where M' is the following TM:

   M'

   Given x:
   1. Simulate M on x
   2. If M halts, accept.

2. Simulate A on $\langle M', w \rangle$
3. If A accepts, accept. If A rejects, reject.

If M halts on w…

- Then M' accepts w
- Therefore, A accepts $\langle M', w \rangle$
- Therefore, H accepts $\langle M, w \rangle$ ✔

If M loops on w…

- Then M' loops on w
- Therefore, A rejects $\langle M', w \rangle$
- Therefore, H rejects $\langle M, w \rangle$ ✔

Some more undecidable problems

- We have seen several interesting examples of undecidable languages
  - SELF-REJECTORS, HALT, PCP, MPCP, ACCEPT
- I'll describe ~~a few more examples~~ one more example
- Each can be proven undecidable via reduction from HALT
- But we will not do the proofs
- (This material will not be on exercises or exams)

Hilbert's 10^{th} problem

- **Informal problem statement:** Given a polynomial equation with integer coefficients such as

$$x^2 + 3xz + y^3 + z^2x^2 = 4xy^2 + 6yz + 2,$$

determine whether there is an integer solution

- **As a language:** $\mathrm{HILBERT10} = \{\langle p, q \rangle : \exists \vec{x} \text{ such that } p(\vec{x}) = q(\vec{x})\}$

Theorem: $\mathrm{HILBERT10}$ is undecidable