Analysis of Boolean Functions

Autumn 2025

Course Summary & Review

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Fourier expansion

- **Fourier Expansion Theorem:** Every $f\colon\{\pm 1\}^n \to \mathbb{R}$ can be uniquely written as a multilinear polynomial

$$f(x) = \sum_{S\subseteq[n]} \hat{f}(S) \cdot \prod_{i\in S} x_i$$

- Character function: $\chi_S(x) = \prod_{i\in S} x_i$

Inner product space of functions

- **Inner product:** $\langle f, g\rangle = \mathbb{E}_x[f(x) \cdot g(x)]$
- **Plancherel's Theorem:** $\langle f, g\rangle = \sum_{S \subseteq [n]} \hat{f}(S) \cdot \hat{g}(S)$
- **Fourier Coefficient Formula:** $\hat{f}(S) = \mathbb{E}_x[f(x) \cdot \chi_S(x)]$
- **Parseval's Theorem:** $\sum_{S \subseteq [n]} \hat{f}(S)^2 = \mathbb{E}_x[f(x)^2]$

Interpreting the Fourier spectrum

- $\mathbb{E}[f] = \hat{f}(\emptyset)$
- $\mathrm{Var}[f] = \sum_{S \neq \emptyset} \hat{f}(S)^2$
- If $f\colon \{\pm 1\}^n \to \{\pm 1\}$, then $\hat{f}(S) = 1 - 2 \cdot \mathrm{dist}(f, \chi_S)$

Complexity measures

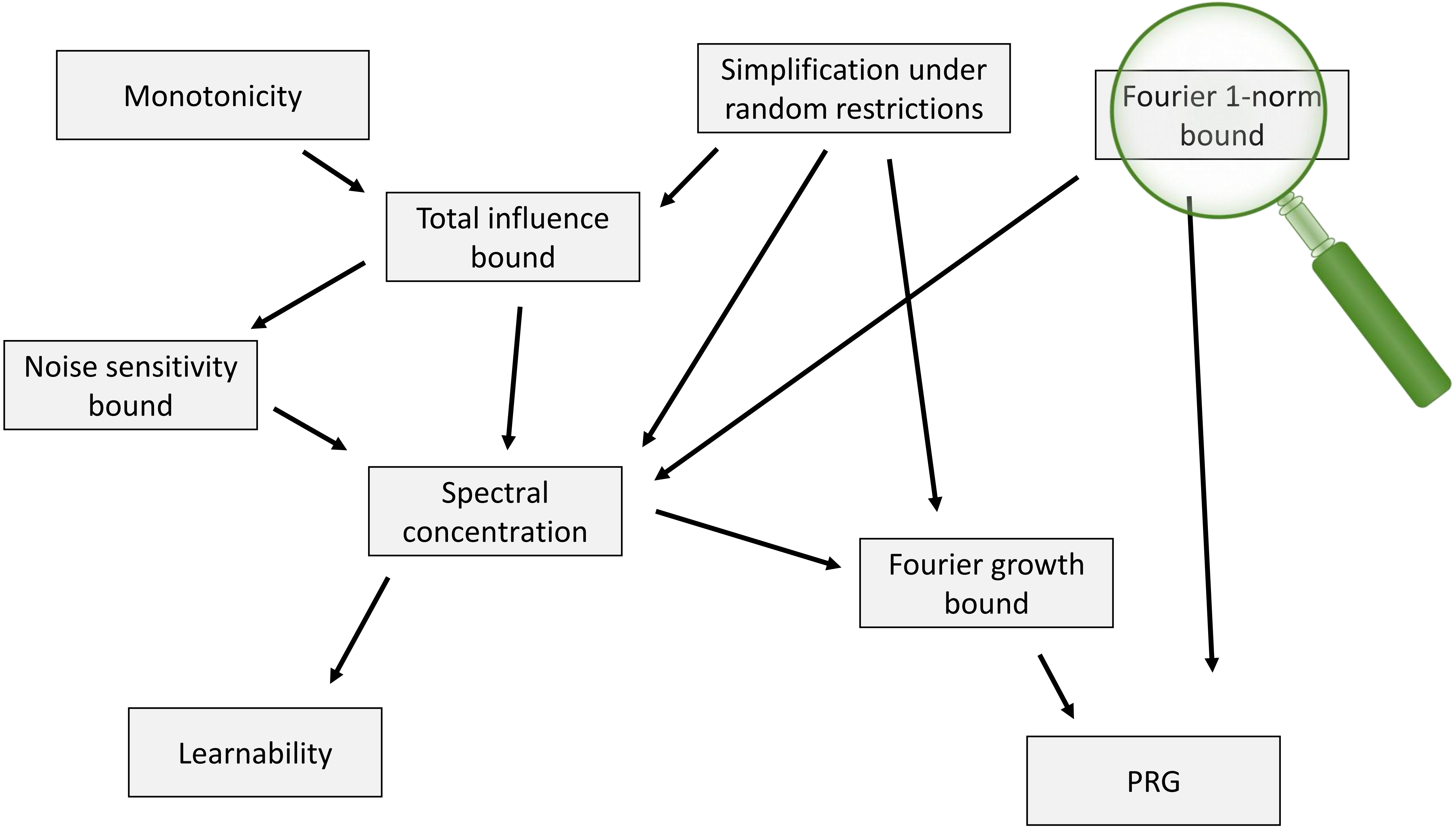
- How difficult is it to compute $f(x)$?
  - Decision tree depth/size
  - AC^0 circuit depth/size
  - ROBP width
  - PTF sparsity

- How complicated is the Fourier expansion of f?
  - $\deg(f)$
  - $\|\hat{f}\|_1$
  - Spectral concentration
  - $L_{1,k}(f)$

- How much time does is take to learn f?
  - From random examples
  - From queries

- How sensitive is f to bit flips?
  - Total influence
  - Noise sensitivity
  - Random restrictions
  - Juntas
  - Monotone/unate functions

- How many random bits does it take to fool f?

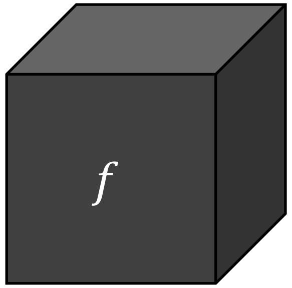

Monotonicity
Simplification under random restrictions
Fourier 1-norm bound
Total influence bound
Noise sensitivity bound
Spectral concentration
Fourier growth bound
Learnability
PRG

Fourier 1-norm: $\hat{\|} f \hat{\|}_1 = \sum_S |\hat{f}(S)|$

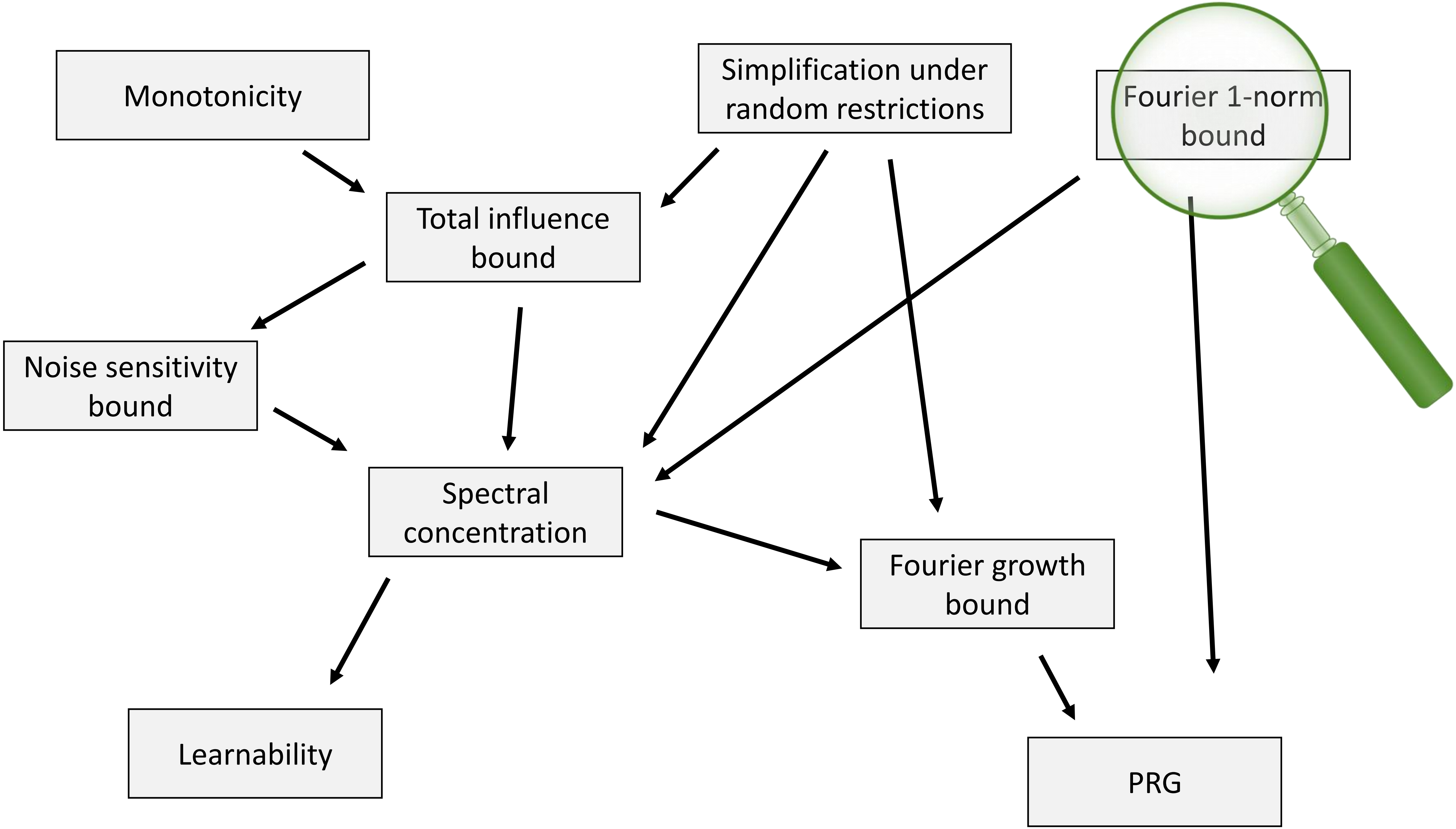
Model	**Bound on $\hat{\|} f \hat{\|}_1$**	**Proof technique**
Size-s decision trees	s	Decompose as sum over leaves
Size-s AND/OR/XOR formulas	$2^{O(s)}$	Induction
Boolean k-junta	$2^{k/2}$	Cauchy-Schwarz

- **Theorem:** Can fool f using a seed of length $O\left(\log\left(n \cdot \hat{\|} f \hat{\|}_1 / \varepsilon\right)\right)$
- **Theorem:** f is ε-concentrated on a set of $\hat{\|} f \hat{\|}_1^2 / \varepsilon$ Fourier coefficients

Concentration ⇒ Learnability



- **Theorem** (Linial-Mansour-Nisan): If f is ε-concentrated up to degree k, then f can be learned from random examples in time $n^{O(k)} \cdot \mathrm{poly}(1/\varepsilon)$
- **Theorem** (Goldreich-Levin): Given query access to f, we can find all S such that $\left|\hat{f}(S)\right| \geq \theta$ in time $\mathrm{poly}(n/\theta)$
- **Theorem** (Kushilevitz-Mansour): If f is ε-concentrated on a set of M Fourier coefficients, then f can be learned from queries in time $\mathrm{poly}(n, M, 1/\varepsilon)$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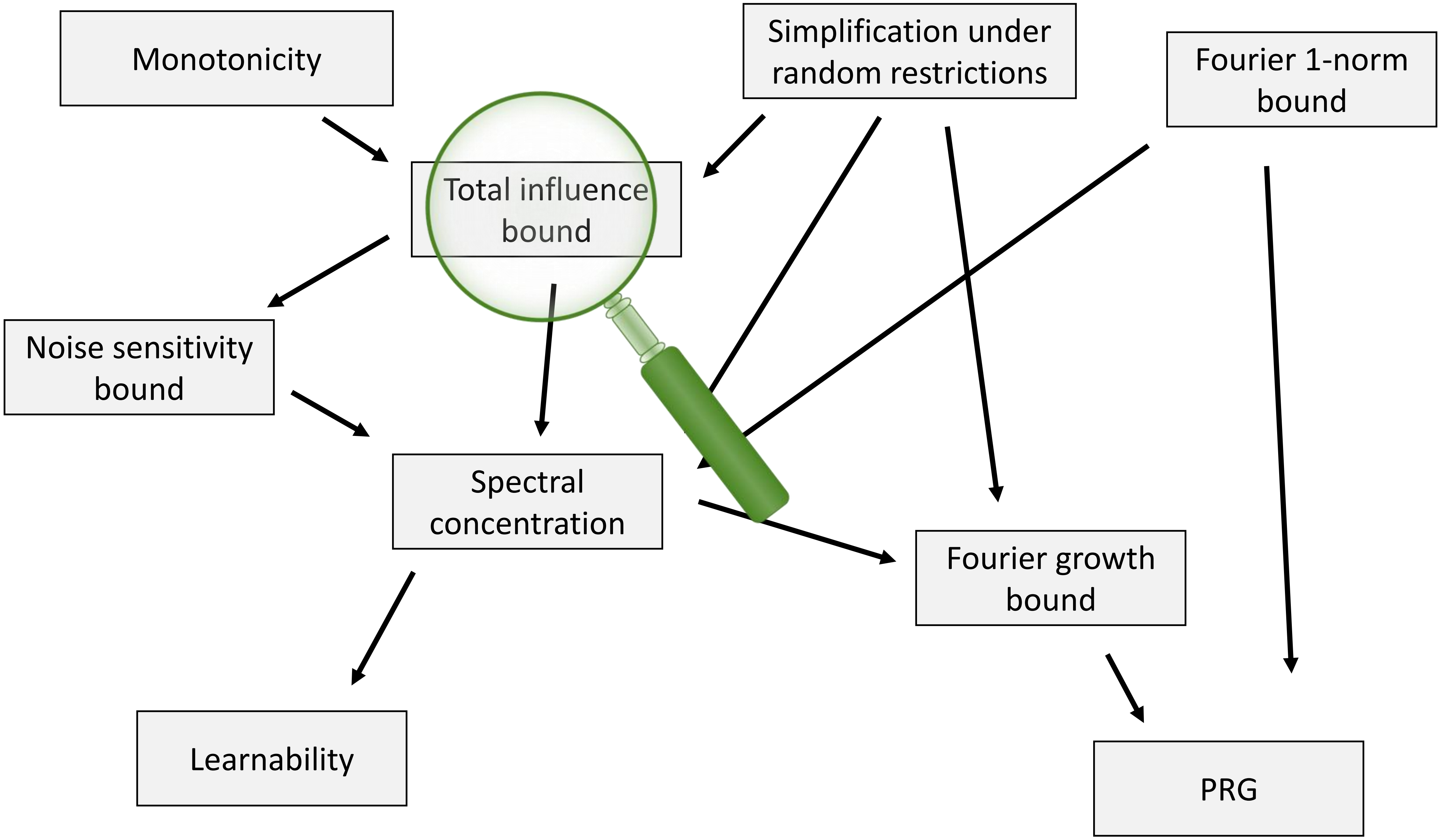
$$\mathrm{I}[f] = \sum_{i=1}^{n} \mathrm{Inf}_i[f] = \mathbb{E}_x\left[\mathrm{sens}_f(x)\right] = \mathbb{E}_{S\sim\mathcal{S}_f}[|S|]$$

Model	Bound on $\mathrm{I}[f]$	Proof technique
Size-s decision trees	$\log s$	Sensitivity
Width-w DNFs	$O(w)$	Sensitivity
Unate functions	$\sqrt{n}$	Level-1 Fourier coefficients
Unate size-s decision trees	$\sqrt{\log s}$	Level-1 Fourier coefficients
Size-s AC_d^0 circuits	$O(\log s)^{d-1}$	Random restrictions
Size-s De Morgan formulas	$O(\sqrt{s})$	Random restrictions
Degree-k PTFs	$O\left(n^{1-2^{-k}}\right)$	Derivatives



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Noise sensitivity

$x = 1000101110100001111010111001011111011110$
$y = 1000101010100000111010111011011111011100$

- $\mathrm{NS}_\delta[f] = \Pr[f(x) \neq f(y)]$, where x is uniform and $y_i = -x_i$ with prob δ

Model	**Bound on $\mathrm{NS}_\delta[f]$**
LTFs	$O(\sqrt{\delta})$
Degree-k PTFs	$O\left(\delta^{2^{-k}}\right)$

- **Theorem:** f is $O\left(\mathrm{NS}_{1/k}[f]\right)$-concentrated up to degree k

Hypercontractivity

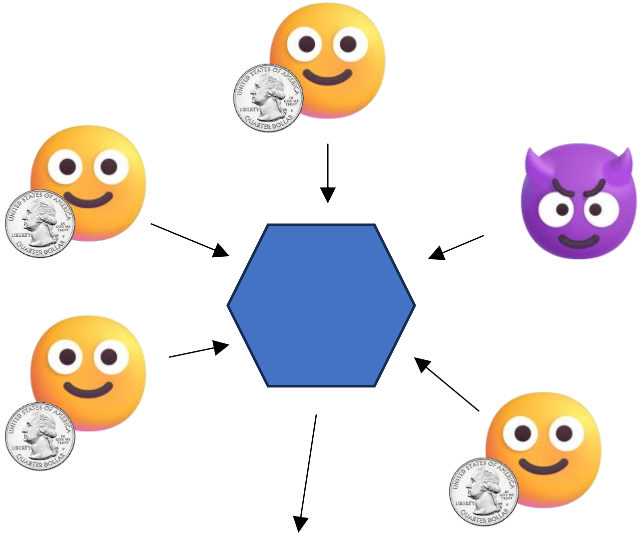
$x =$ 10001011101000011110101110010111110111110
$y =$ 10001010101000001110101110110111110111000

- x and y are ρ-correlated if $\mathbb{E}[x_i] = \mathbb{E}[y_i] = 0$ and $\mathbb{E}[x_i y_i] = \rho$ independently for each i
- **Hypercontractivity Theorem:** $\mathbb{E}[f(x) \cdot g(y)] \leq \|f\|_{1+r} \cdot \|g\|_{1+\rho^2/r}$
- Alternate form: $\left\|T_\rho f\right\|_{1+r} \leq \|f\|_{1+\rho^2 \cdot r}$
- Corollaries in terms of degree, e.g., $\|f\|_2 \leq 2^{O(\deg(f))} \cdot \|f\|_1$

Friedgut's junta theorem

- Let $f\colon \{\pm 1\}^n \to \{\pm 1\}$
- **Friedgut's Junta Theorem:** f is close to a k-junta where $k = 2^{O(\mathrm{I}[f])}$

Collective coin flipping

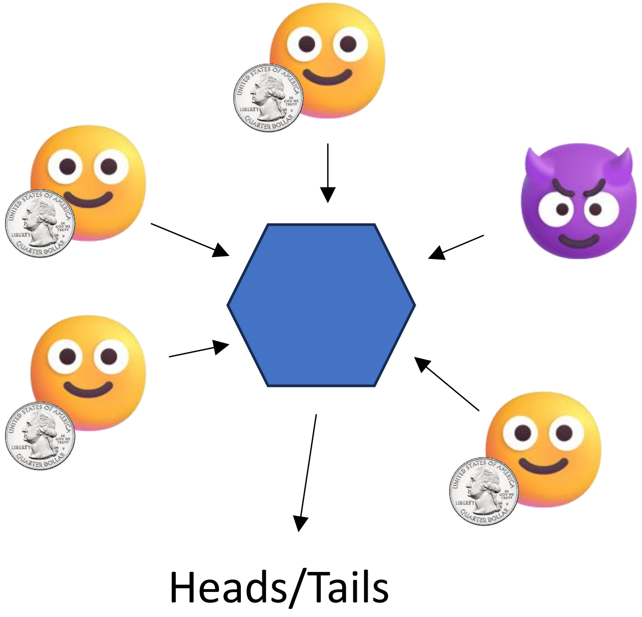


- Let $f\colon \{\pm 1\}^n \to \{\pm 1\}$
- **Poincaré Inequality:** $\mathrm{I}[f] \geq \mathrm{Var}[f]$
- **Kahn-Kalai-Linial Theorem:** There exists i such that

$$\mathrm{Inf}_i[f] \geq \Omega\left(\mathrm{Var}[f] \cdot \frac{\log n}{n}\right)$$

- **Corollary:** If f is near-balanced, then f is not resilient against μn cheaters

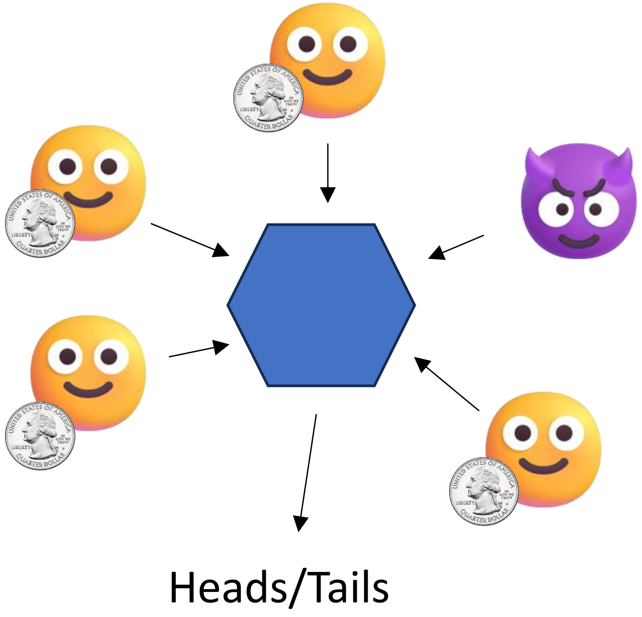
Collective coin flipping



- KKL is optimal due to Tribes. However…
- Let $f\colon \{\pm 1\}^n \to \{\pm 1\}$
- Suppose f can be computed by a size-s decision tree
- **O'Donnell-Saks-Schramm-Servedio Inequality:** There exists i such that

$$\mathrm{Inf}_i[f] \geq \frac{\mathrm{Var}[f]}{\log s}$$

Collective coin flipping



Hypercontractivity

$\Downarrow$

Friedgut's Junta Theorem

$\Downarrow$

KKL Theorem

$\Downarrow$

Limitations of Resilient Functions

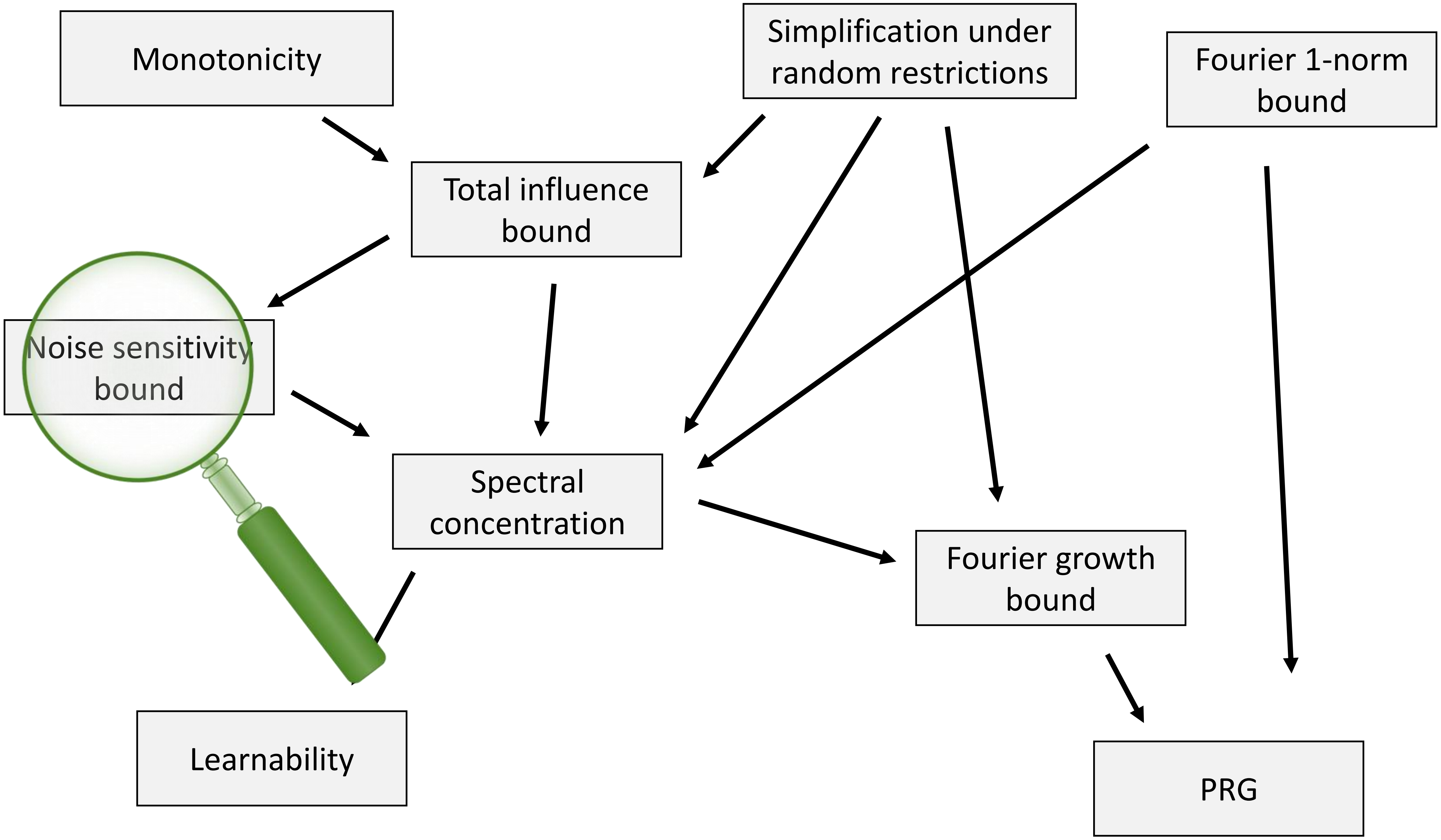
OSSS Inequality

Arrow's theorem

- Let $f\colon \{\pm 1\}^n \to \{\pm 1\}$
- **Arrow's Theorem:** Assume that if we use f to run pairwise elections, Condorcet's paradox never occurs. Then $\pm f$ is a dictator.
- **FKN Theorem:** If $W^1[f] \geq 1 - \varepsilon$, then $\pm f$ is $O(\varepsilon)$-close to a dictator.
- **Robust Arrow's Theorem:** Assume that if we use f to run pairwise elections, Condorcet's paradox rarely occurs. Then $\pm f$ is close to a dictator.

Property testing

- Suppose we have query access to unknown $f: \{\pm 1\}^n \to \{\pm 1\}$
- **Linearity Testing Theorem:** Using 3 queries, we can distinguish the case $f = \chi_S$ from the case that f is far from every χ_S
- **Dictator Testing Theorem:** Using 3 queries, we can distinguish the case $f = \chi_{\{i\}}$ from the case that f is far from every $\chi_{\{i\}}$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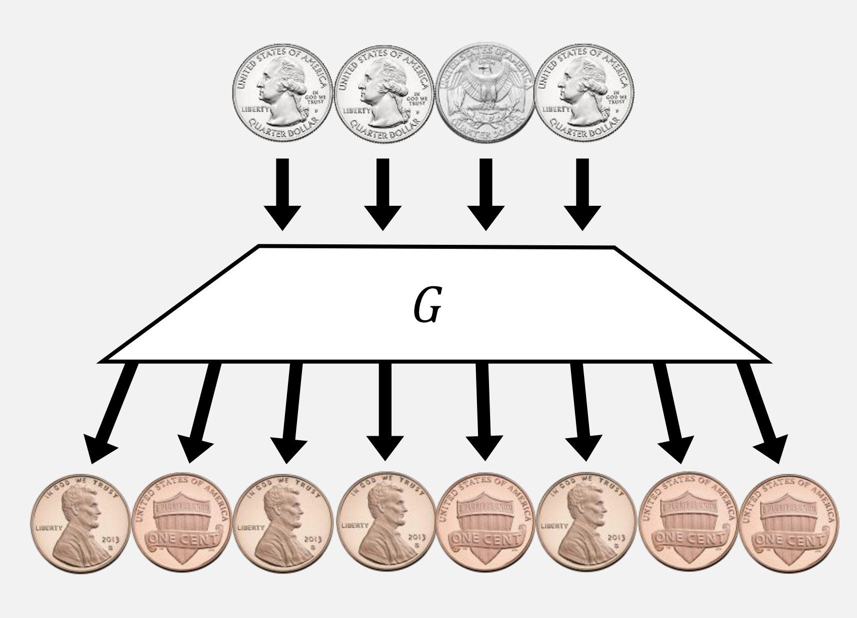
Fourier growth: $L_{1,k}(f) = \sum_{|S|=k} |\hat{f}(S)|$

Model	**Bound on $L_{1,k}(f)$**	**Proof technique**
Size-s AC^0_d circuits	$O(\log s)^{(d-1)\cdot k}$	Random restrictions
Width-w regular oblivious ROBPs	w^k	Induction, local monotonization
Depth-d decision trees	$\binom{d}{k}$	[multiple]

- Note: Fourier growth bounds do not imply spectral concentration
- This can be a good thing!

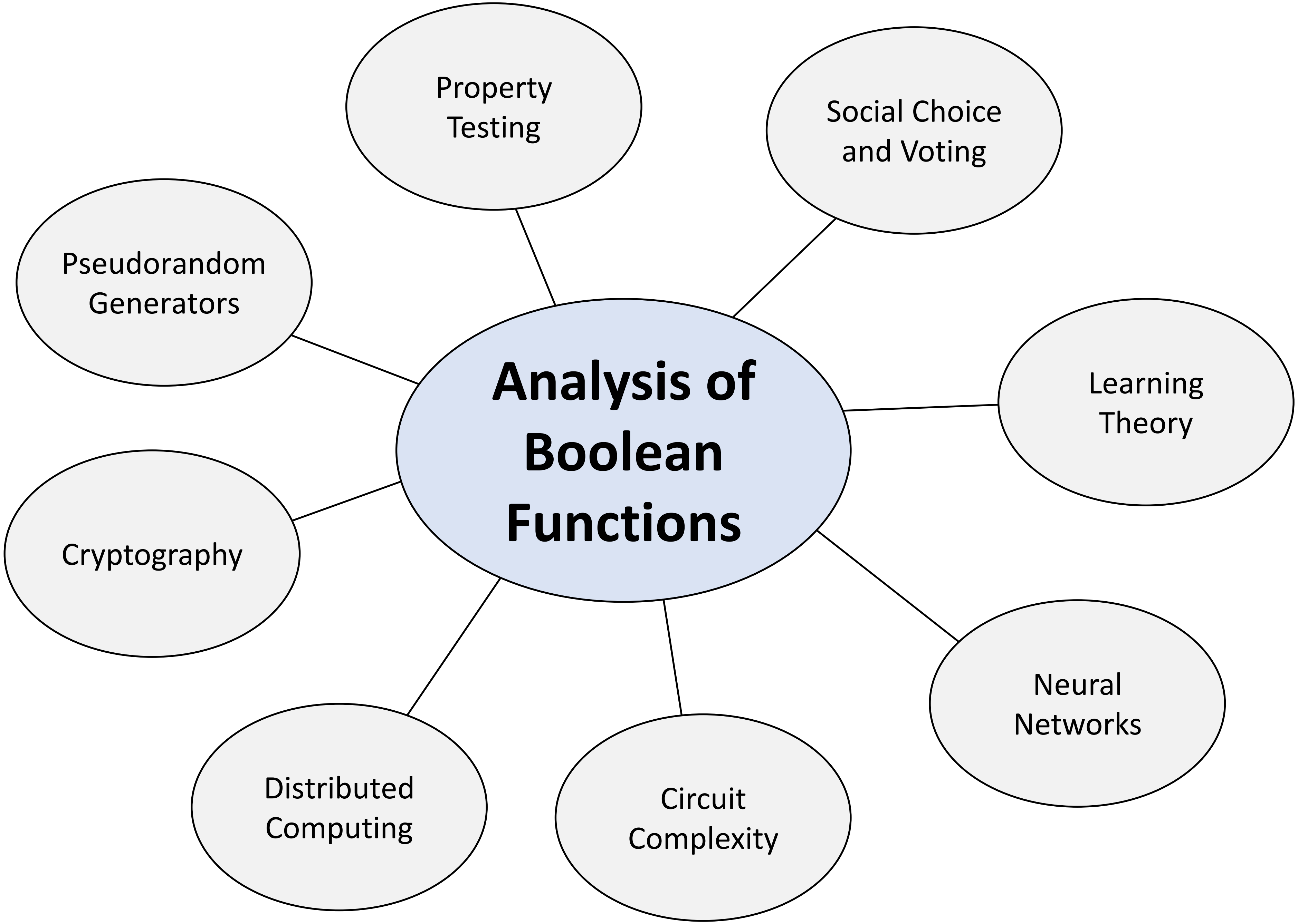
Fourier growth bound ⇒ PRG



- Let $\mathcal{F}$ be a class of Boolean functions that is closed under restrictions
- Assume that for every $f \in \mathcal{F}$ and every k, we have $L_{1,k}(f) \leq b^k$
- **Theorem:** Can fool $\mathcal{F}$ using a seed of length $\tilde{O}(b^2 \cdot \log(n/\varepsilon) \cdot \log(1/\varepsilon))$

A few of the many topics we didn't discuss

- Gaussian space and the invariance principle
- p-biased Fourier analysis
- Threshold phenomena
- Expansion of noisy hypercube


Analysis of Boolean Functions
Property Testing
Social Choice and Voting
Learning Theory
Neural Networks
Circuit Complexity
Distributed Computing
Cryptography
Pseudorandom Generators

Thank you!

- Being your instructor has been a privilege
- I look forward to reading your expositions
- Please fill out the Graduate Course Feedback Form using My.UChicago (deadline is Sunday, December 14)