Exercises 3 & 4

Analysis of Boolean Functions, Autumn 2025, University of Chicago
Instructor: William Hoza (williamhoza@uchicago.edu)

Submission. Solutions are due **Friday, October 17** at 11:59pm Central time. Submit your solutions through Gradescope. You are encouraged, but not required, to typeset your solutions using a LaTeX editor such as Overleaf.

The policies below can also be found on the course webpage.

Collaboration. You are encouraged to collaborate with your classmates on exercises, but you must adhere to the following rules.

- Work on each exercise on your own for at least five minutes before discussing it with classmates.
- Feel free to explain your ideas to your classmates in person, and feel free to use whiteboards/chalkboards/etc. However, do not share any written/typeset solutions with your classmates for them to study on their own time. This includes partial solutions.
- Write your solutions on your own. While you are writing your solutions, do not consult any notes that you might have taken during discussions with classmates.
- In your write-up, list any classmates who helped you figure out the solution. The fact that student A contributed to student B's solution does not necessarily mean that student B contributed to student A's solution.

Permitted Resources for Full Credit. In addition to discussions with me and discussions with classmates as discussed above, you may also use the course textbook, any slides or notes posted in the "Course Timeline" section of the course webpage, and Wikipedia. If you wish to receive full credit on an exercise, you may not use any other resources.

Outside Resources for Partial Credit. If you wish, you may use outside resources (ChatGPT, Stack Exchange, etc.) to solve an exercise for partial credit. If you decide to go this route, you must make a note of which outside resources you used when you were working on each exercise. You must disclose using a resource even if it was ultimately unhelpful for solving the exercise. Furthermore, if you consult an outside resource while working on an exercise, then you must not discuss that exercise with your classmates.

As a reminder, the standard and Fourier p-norm of $f\colon \{\pm 1\}^n \to \mathbb{R}$ are defined by

$$\|f\|_p = \mathop{\mathbb{E}}_{x \in \{\pm 1\}^n}[|f(x)|^p]^{1/p} \qquad\qquad \hat{\|} f \hat{\|}_p = \left(\sum_{S \subseteq [n]} |\widehat{f}(S)|^p \right)^{1/p}.$$

Exercise 3 (10 points)**.**

(a) Let $f\colon \{0,1\}^n \to \mathbb{R}$. Prove that $\hat{\|} f \hat{\|}_1 \le 2^{n/2} \cdot \|f\|_2$.

(b) For each even positive integer $n \in \mathbb{N}$, the *inner product modulo 2* function $\mathsf{IP}_n\colon \{0,1\}^n \to \{\pm 1\}$ is defined by

$$\mathsf{IP}_n(x,y) = (-1)^{\sum_{i=1}^{n/2} x_i y_i}.$$

Prove that $|\widehat{\mathsf{IP}_n}(S)| = 2^{-n/2}$ for every $S \subseteq [n]$, hence $\hat{\|} \mathsf{IP}_n \hat{\|}_1 = 2^{n/2}$, demonstrating that part (a) is tight.

(c) For each $k \in \mathbb{N}$, the *address function* $\mathsf{ADDR}_k\colon \{0,1\}^{2^k} \times \{0,1\}^k \to \{\pm 1\}$ is given by $\mathsf{ADDR}(x,i) = (-1)^{x_{i+1}}$. Here i is a number in $\{0,1,2,\ldots,2^k-1\}$ represented in binary. Prove that $\hat{\|} \mathsf{ADDR}_k \hat{\|}_1 \ge 2^k$.

(Note that ADDR_k can be computed by a decision tree of depth $k+1$, hence size at most 2^{k+1}. In class, we showed that if $f\colon \{0,1\}^n \to \{\pm 1\}$ is computable by a size-s decision tree, then $\hat{\|} f \hat{\|}_1 \le s$. The address function is an example where the bound is tight up to a factor of two.)

Exercise 4 (10 points)**.**

(a) Let $f\colon \{\pm 1\}^n \to \mathbb{R}$. Prove that $\|f\|_\infty \leq \hat{\|} f \hat{\|}_1$.

(b) Let $g, h\colon \{\pm 1\}^n \to \mathbb{R}$ and let $f(x) = g(x) \cdot h(x)$. Prove that $\hat{\|} f \hat{\|}_1 \leq \hat{\|} g \hat{\|}_1 \cdot \hat{\|} h \hat{\|}_1$.

(c) Let $g\colon \{\pm 1\}^m \to \mathbb{R}$, let $h_1, \dots, h_m\colon \{\pm 1\}^n \to \{\pm 1\}$, and let $f(x) = g(h_1(x), \dots, h_m(x))$. Prove that

$$\hat{\|} f \hat{\|}_1 \leq \hat{\|} g \hat{\|}_1 \cdot \prod_{i=1}^{m} \hat{\|} h_i \hat{\|}_1.$$

(d) Let $f\colon \{\pm 1\}^n \to \{\pm 1\}$ be a function that is computable by a formula consisting of s AND, OR, and XOR gates with unbounded fan-in.

That is, f is computed by a tree with s internal nodes. Each leaf is labeled with a variable x_i where $i \in [n]$, a negated variable $-x_i$, or a constant $b \in \{\pm 1\}$. Each internal node is labeled with AND, OR, or XOR. Given an input $x \in \{\pm 1\}^n$, each internal node applies its label-function to its children, until eventually the root node produces the output $f(x)$. We emphasize that an internal node is permitted to have more than two children. Note also that since we encode bits using ± 1, the "AND" function is actually max, the "OR" function is actually min, and the "XOR" function is actually multiplication.

Prove that $\hat{\|} f \hat{\|}_1 \leq 2^{O(s)}$.