Exercises 1 & 2

Analysis of Boolean Functions, Autumn 2025, University of Chicago
Instructor: William Hoza (williamhoza@uchicago.edu)

Submission. Solutions are due **Friday, October 10** at 11:59pm Central time. Submit your solutions through Gradescope. You are encouraged, but not required, to typeset your solutions using a LaTeX editor such as Overleaf.

The policies below can also be found on the course webpage.

Collaboration. You are encouraged to collaborate with your classmates on exercises, but you must adhere to the following rules.

- Work on each exercise on your own for at least five minutes before discussing it with classmates.
- Feel free to explain your ideas to your classmates in person, and feel free to use whiteboards/chalkboards/etc. However, do not share any written/typeset solutions with your classmates for them to study on their own time. This includes partial solutions.
- Write your solutions on your own. While you are writing your solutions, do not consult any notes that you might have taken during discussions with classmates.
- In your write-up, list any classmates who helped you figure out the solution. The fact that student A contributed to student B's solution does not necessarily mean that student B contributed to student A's solution.

Permitted Resources for Full Credit. In addition to discussions with me and discussions with classmates as discussed above, you may also use the course textbook, any slides or notes posted in the "Course Timeline" section of the course webpage, and Wikipedia. If you wish to receive full credit on an exercise, you may not use any other resources.

Outside Resources for Partial Credit. If you wish, you may use outside resources (ChatGPT, Stack Exchange, etc.) to solve an exercise for partial credit. If you decide to go this route, you must make a note of which outside resources you used when you were working on each exercise. You must disclose using a resource even if it was ultimately unhelpful for solving the exercise. Furthermore, if you consult an outside resource while working on an exercise, then you must not discuss that exercise with your classmates.

Exercise 1 (10 points)**.** Let $f\colon \{0,1\}^n \to \mathbb{R}$.

(a) Prove that f can be written as a multilinear polynomial: $f(x) = \sum_{S \subseteq [n]} \widetilde{f}(S) \cdot \prod_{i \in S} x_i$.

Note: We are working with $\{0,1\}$-valued variables, so this is not the Fourier expansion of f.

(b) Prove that the representation in part (a) is unique.

Hint: Start with the case $f(x) \equiv 0$. Prove $\widetilde{f}(S) = 0$ by strong induction on $|S|$. Use the indicator vector of S.

(c) Prove that $\max\left\{|S| : \widetilde{f}(S) \neq 0\right\} = \max\left\{|S| : \widehat{f}(S) \neq 0\right\}$, hence we can use the notation $\deg(f)$ to refer to either quantity.

Hint: Think about basis functions first.

(d) Assume f is integer-valued. Prove that every Fourier coefficient $\widehat{f}(S)$ is an integer multiple of $2^{-\deg(f)}$.

Hint: Come up with a formula for $\widehat{f}(S)$ in terms of the $\widetilde{f}(T)$ coefficients.

In class, we design algorithms for linearity testing and dictator testing. In this exercise, you will improve the constant factors in the analyses.

Exercise 2 (10 points)**.**

(a) Let $f\colon \mathbb{F}_2^n \to \{\pm 1\}$. Assume that f is δ-close to χ_S for some $S \subseteq [n]$. Prove that for every $T \neq S$, we have $|\widehat{f}(T)| \leq 2\delta$.

(b) Let $f\colon \mathbb{F}_2^n \to \{\pm 1\}$. Assume that $\Pr_{x,y}[f(x+y) = f(x) \cdot f(y)] \geq 1 - \varepsilon$. Prove that f is δ-close to a character function, where $\delta = \varepsilon/3 + O(\varepsilon^2)$.

(c) Design a dictator testing algorithm with the following properties.

- The algorithm makes three queries to the unknown function $f\colon \{\pm 1\}^n \to \{\pm 1\}$.
- If f is a dictator function, then the algorithm accepts with probability 1.
- There is some constant ε_0 (independent of n) such that for every $\varepsilon \leq \varepsilon_0$, if the algorithm accepts with probability at least $1 - \varepsilon$, then f is (0.34ε)-close to a dictator function.