Exercises 11-14 [Typo corrected 2024-11-19]

Circuit Complexity, Autumn 2024, University of Chicago
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Submission. Solutions are due **Wednesday, November 20** at 5pm Central time. Submit your solutions through Gradescope. You are encouraged, but not required, to typeset your solutions using a LaTeX editor such as Overleaf.

The policies below can also be found on the course webpage.

Collaboration. You are encouraged to collaborate with your classmates on homework, but you must adhere to the following rules.

- Work on each exercise on your own for at least fifteen minutes before discussing it with your classmates.
- Feel free to explain your ideas to your classmates in person, and feel free to use whiteboards/chalkboards/etc. However, do not share any written/typeset solutions with your classmates for them to study on their own time. This includes partial solutions.
- Write your solutions on your own. While you are writing your solutions, do not consult any notes that you might have taken during discussions with classmates.
- In your write-up, list any classmates who helped you figure out the solution. The fact that student A contributed to student B's solution does not necessarily mean that student B contributed to student A's solution.

Permitted Resources for Full Credit. In addition to discussions with me and discussions with classmates as discussed above, you may also use any slides or notes posted in the "Course Timeline" section of the course webpage, and you may also use Wikipedia. If you wish to receive full credit on an exercise, you may not use any other resources.

Outside Resources for Partial Credit. If you wish, you may use outside resources (ChatGPT, Stack Exchange, etc.) to solve an exercise for partial credit. If you decide to go this route, you must make a note of which outside resources you used when you were working on each exercise. You must disclose using a resource even if it was ultimately unhelpful for solving the exercise. Furthermore, if you consult an outside resource while working on an exercise, then you must not discuss that exercise with your classmates.

For a function $f\colon \{0,1\}^n \to \{0,1\}$, let $\mathsf{CC}(f)$ denote the circuit complexity of f, i.e., the size the minimum circuit that computes f. For this exercise, we are working with unbounded-depth circuits over the full binary basis. Each vertex with fan-in zero is labeled with a variable or a constant.

Exercise 11 (10 points)**.** Define $f\colon \{0,1\}^n \to \{0,1\}$ by $f(x) = 1$ if and only if $x_1 + x_2 + \cdots + x_n \geq 2$.

(a) Assume $n \geq 3$. Prove that there exists $\rho \in \{0, \star\}^n$ such that $|\rho^{-1}(0)| = 1$ and $\mathsf{CC}(f|_\rho) \leq \mathsf{CC}(f) - 2$.

Hint: Is it possible to compute f using a circuit of the form $C(x) = C'(\phi(x_1, x_2), x_3, x_4, \ldots, x_n)$ where $\phi\colon \{0,1\}^2 \to \{0,1\}$?

(b) Prove that $\mathsf{CC}(f) \geq 2n - O(1)$.

For a function $f\colon \{0,1\}^n \to \{0,1\}$, let $\overline{L}(f)$ denote the minimum leafsize of any formula computing f over the full binary basis, , i.e., each gate has fan-in two and computes an arbitrary function $\{0,1\}^2 \to \{0,1\}$.

Exercise 12 (10 points)**.** Prove that there exists $f \in \mathsf{P}$ such that $\overline{L}(f) \geq \widetilde{\Omega}(n^2)$.

In class, we used the AC^0 Criticality Theorem to prove an extremely strong bound on the correlation between the parity function and AC^0 circuits. In this exercise, you will prove that the bound is essentially optimal.

Exercise 13 (10 points)**.**

(a) Let $n, d \in \mathbb{N}$ where $d \geq 2$. Assume for simplicity that $n^{1/(d-1)}$ is an integer. Prove that PARITY_n can be computed by an AC^0_d circuit of size $O(n \cdot 2^{n^{1/(d-1)}})$, with an OR gate on top.

Hint: Divide and conquer, using both brute-force CNFs and brute-force DNFs. Watch out for negations.

(b) Let $n, d, t \in \mathbb{N}$ where $d \geq 2$. Assume for simplicity that t is even and $(n/t)^{1/(d-1)}$ is an integer. Prove that there exists an AC^0_d circuit $C\colon \{0,1\}^n \to \{0,1\}$ of size $\cancel{O(n \cdot 2^{(n/t)^{d-1}})}$ $O(n \cdot 2^{(n/t)^{\frac{1}{d-1}}})$ such that

$$\Pr_{x \in \{0,1\}^n}[C(x) = \mathsf{PARITY}_n(x)] \geq \frac{1}{2} + 2^{-t}.$$

Hint: Use part (a).

In class, we discuss the AC^0 Criticality Theorem:

Theorem 1 (AC^0 Criticality Theorem)**.** *Let C be a size-S AC^0_d circuit, let $p \in (0,1)$, and let $D \in \mathbb{N}$. Then*

$$\Pr_{\rho \sim R_p}[\mathsf{DTDepth}(C|_\rho) \geq D] \leq (p \cdot O(\log S)^{d-1})^D.$$

We do not prove the AC^0 Criticality Theorem in class. However, we do prove the Switching Lemma, repeated here for convenience.

Lemma 1 (The Switching Lemma)**.** *Let C be a width-w DNF or CNF*[1] *formula, let $p \in (0,1)$, and let $D \in \mathbb{N}$. Then*

$$\Pr_{\rho \sim R_p}[\mathsf{DTDepth}(C|_\rho) \geq D] \leq O(pw)^D.$$

In this exercise, you will use the Switching Lemma to prove the following weaker variant of the AC^0 Criticality Theorem.

Theorem 2. *Let C be a size-S AC^0_d circuit, let $p \in (0,1)$, and let $D \in \mathbb{N}$. Then*

$$\Pr_{\rho \sim R_p}[\mathsf{DTDepth}(C|_\rho) \geq D] \leq (p \cdot O(\log S)^{d-1})^D + \frac{1}{S}.$$

Exercise 14 (10 points)**.**

(a) Let T be a depth-w decision tree. Prove that T can be computed by a width-w DNF, and T can also be computed by a width-w CNF.

(b) Let us define "$\mathsf{AC}^0_d[S] \circ \mathsf{DT}[w]$" to be the class of functions of the form $C(x) = g(h_1(x), \ldots, h_m(x))$, where g is an AC^0_d circuit of size S and $h_1, \ldots, h_m$ are depth-w decision trees. Use the Switching Lemma and part (a) to prove that if $C \in \mathsf{AC}^0_d[S] \circ \mathsf{DT}[w]$ where $d \geq 1$, then for every $D \geq 0$, we have

$$\Pr_{\rho \sim R_p}\left[C|_\rho \notin \mathsf{AC}^0_{d-1}[S] \circ \mathsf{DT}[D]\right] \leq S \cdot O(pw)^{D+1}.$$

(c) Use part (b) to prove Theorem 2.

Hints: Applying a restriction sampled from R_p is equivalent to applying d independent rounds of restrictions, sampled from $R_{p_1}, R_{p_2}, \ldots, R_{p_d}$, such that $p = p_1 \cdot p_2 \cdots p_d$. Under such a sequence of restrictions, argue that the circuit simplifies as follows, for a suitable parameter $w = \Theta(\log S)$:

$$\mathsf{AC}^0_d[S] \circ \mathsf{DT}[1] \to \mathsf{AC}^0_{d-1}[S] \circ \mathsf{DT}[w] \to \mathsf{AC}^0_{d-2}[S] \circ \mathsf{DT}[w] \to \cdots \to \mathsf{AC}^0_1[S] \circ \mathsf{DT}[w] \to \mathsf{AC}^0_0[S] \circ \mathsf{DT}[D-1].$$

Note that an AC^0_1 circuit is simply a conjunction or disjunction of literals, and an AC^0_0 circuit is simply a literal or a constant.

(d) (Optional; 1 point extra credit) Use Theorem 2 to prove that every AC^0_d circuit computing PARITY_n must have size $2^{\Omega(n^{1/(d-1)})}$.

[1] In class, we focused on the case of DNFs, but the CNF case follows by considering $1 - C$.