Exercises 1 & 2

Circuit Complexity, Autumn 2024, University of Chicago
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Submission. Solutions are due **Wednesday, October 9** at 5pm Central time. Submit your solutions through Gradescope. You are encouraged, but not required, to typeset your solutions using a LaTeX editor such as Overleaf.

The policies below can also be found on the course webpage.

Collaboration. You are encouraged to collaborate with your classmates on homework, but you must adhere to the following rules.

- Work on each exercise on your own for at least fifteen minutes before discussing it with your classmates.
- Feel free to explain your ideas to your classmates in person, and feel free to use whiteboards/chalkboards/etc. However, do not share any written/typeset solutions with your classmates for them to study on their own time. This includes partial solutions.
- Write your solutions on your own. While you are writing your solutions, do not consult any notes that you might have taken during discussions with classmates.
- In your write-up, list any classmates who helped you figure out the solution. The fact that student A contributed to student B's solution does not necessarily mean that student B contributed to student A's solution.

Permitted Resources for Full Credit. In addition to discussions with me and discussions with classmates as discussed above, you may also use any slides or notes posted in the "Course Timeline" section of the course webpage, and you may also use Wikipedia. If you wish to receive full credit on an exercise, you may not use any other resources.

Outside Resources for Partial Credit. If you wish, you may use outside resources (ChatGPT, Stack Exchange, etc.) to solve an exercise for partial credit. If you decide to go this route, you must make a note of which outside resources you used when you were working on each exercise. You must disclose using a resource even if it was ultimately unhelpful for solving the exercise. Furthermore, if you consult an outside resource while working on an exercise, then you must not discuss that exercise with your classmates.

A *DNF*[1] *formula* is an "OR of ANDs of literals." That is, it is an expression of the form $\bigvee_{i=1}^{m} \bigwedge_{j=1}^{w_i} \ell_{ij}$, where each ℓ_{ij} is a literal (i.e., a variable or its negation). Each "AND of literals" $\bigwedge_{j=1}^{w_i} \ell_{ij}$ is called a *term*. The *size* of the DNF formula is the number of terms (m).

Exercise 1 (10 points)**.** Let $n \in \mathbb{N}$.

(a) Construct a DNF formula of size $\binom{n}{\lceil n/2 \rceil}$ that computes MAJ_n. Briefly explain your construction.

(b) Prove that every DNF formula computing MAJ_n has size at least $\binom{n}{\lceil n/2 \rceil}$.

Note: $\binom{n}{\lceil n/2 \rceil} = \Theta(2^n/\sqrt{n})$, so the exercise shows that the majority function requires very large DNF formulas.

[1] Disjunctive Normal Form

Exercise 2 (10 points)**.** Let $n \in \mathbb{N}$.

(a) Let $C\colon \{0,1\}^n \to \{0,1\}$ be a circuit of size s, and let $y \in \{0,1\}^n$. Construct a circuit C' of size $s + O(n)$ computing the following function:

$$C'(x) = \begin{cases} 1 & \text{if } x = y \\ C(x) & \text{if } x \neq y. \end{cases}$$

Briefly explain your construction.

(b) Let $s_{\max}$ be the maximum circuit complexity of any function $f\colon \{0,1\}^n \to \{0,1\}$. In class, we prove that $s_{\max} = \Theta(2^n/n)$. Prove that for every $s < s_{\max}$, there exists a function $f\colon \{0,1\}^n \to \{0,1\}$ such that f cannot be computed by a circuit of size s, but f can be computed by a circuit of size $s + O(n)$.

Hint for part (b): Construct a sequence of functions, beginning with a trivial function and ending with a maximally-hard function. Then apply part (a).