Simple Optimal Hitting Sets for Small-Success **RL**

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The University of Texas at Austin

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[2] Supported by NSF Grant CCF-1526952, NSF Grant CCF-1705028, and a Simons Investigator Award (#409864)

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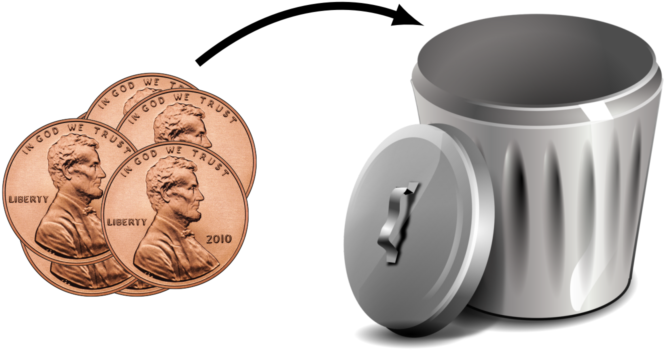

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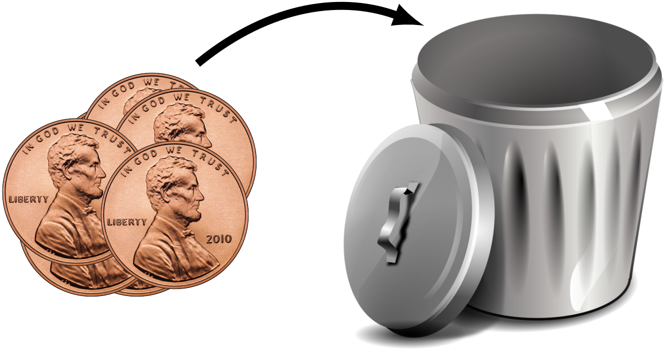
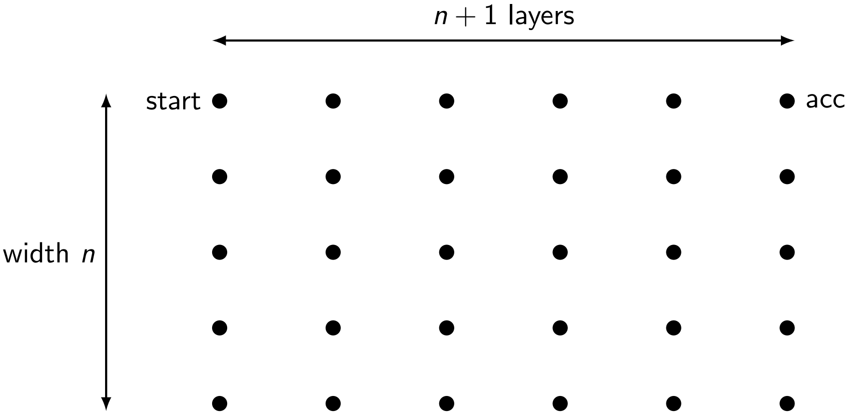


Read-once branching programs



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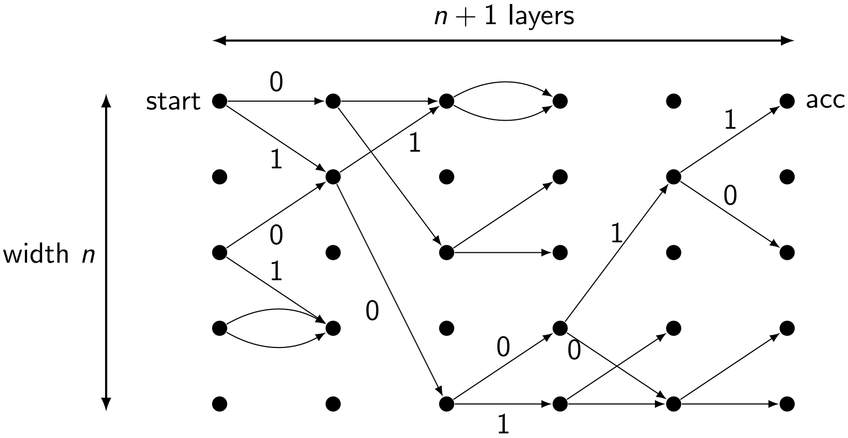

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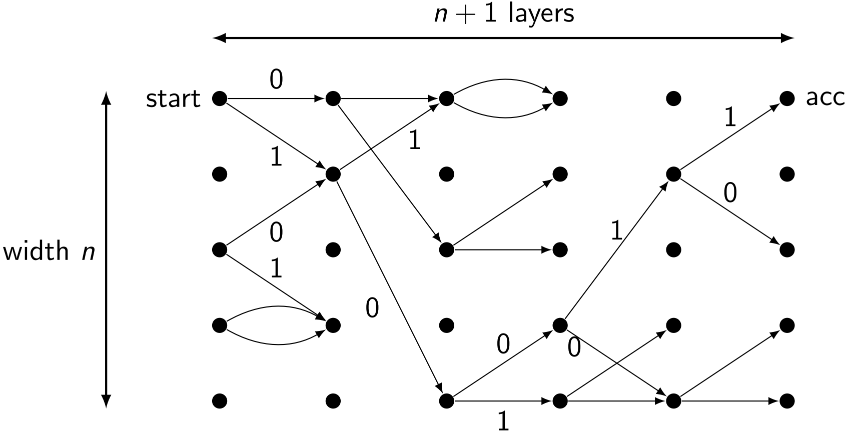

$x =$

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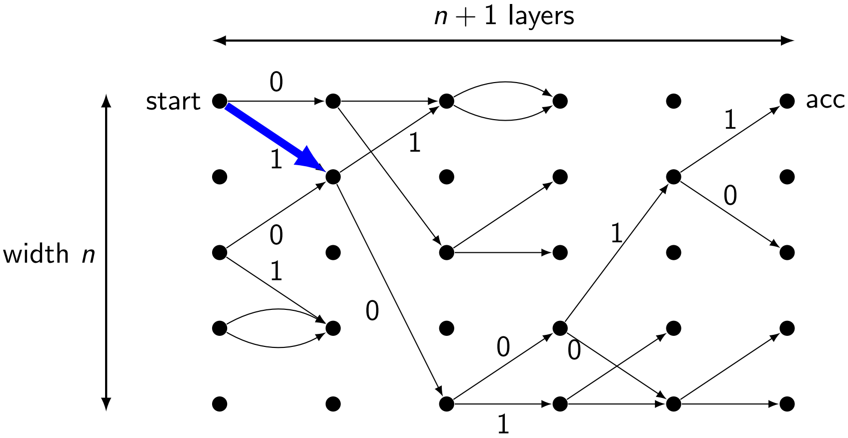

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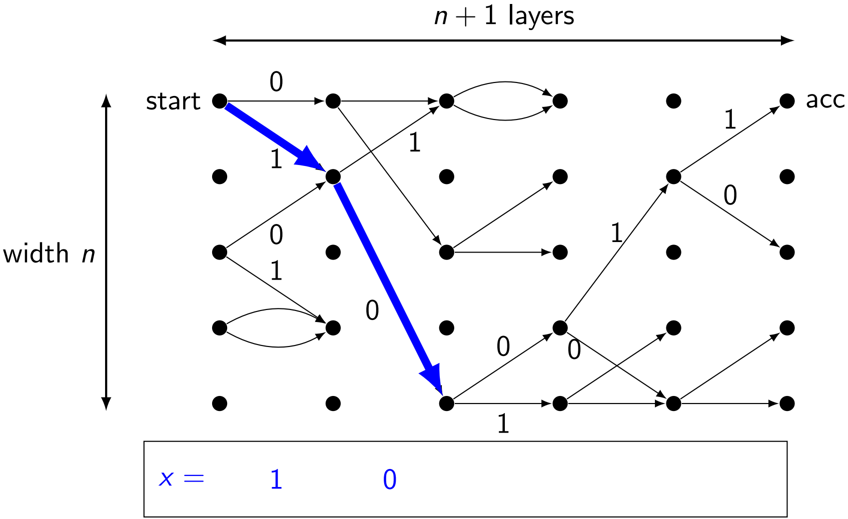

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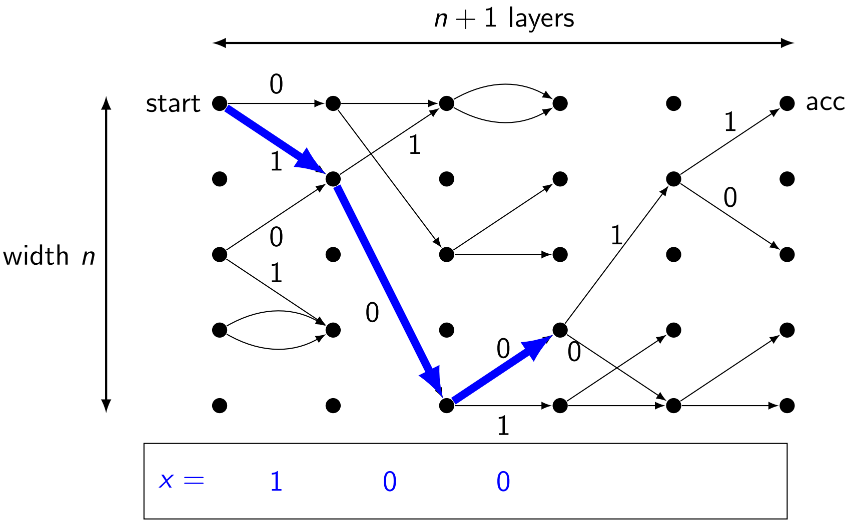

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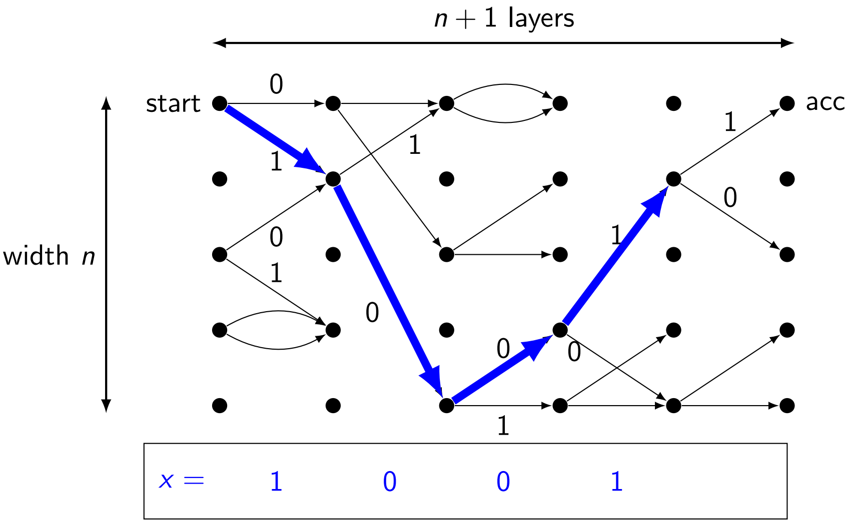

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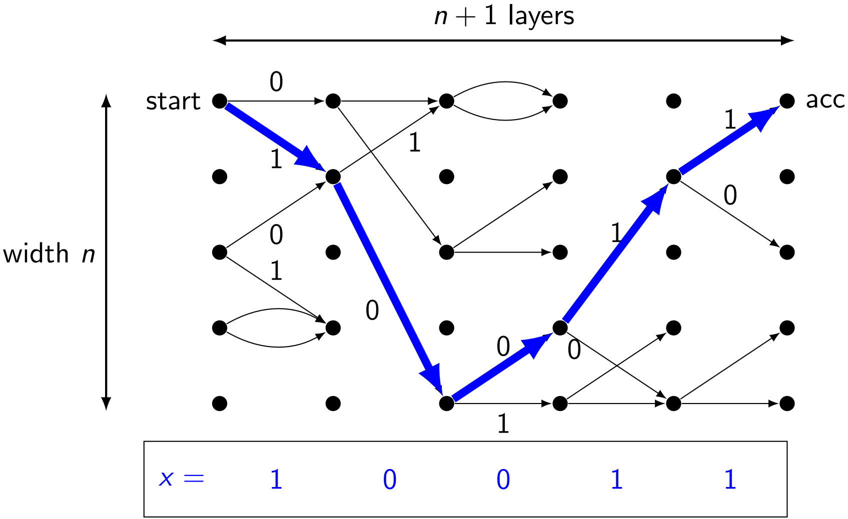

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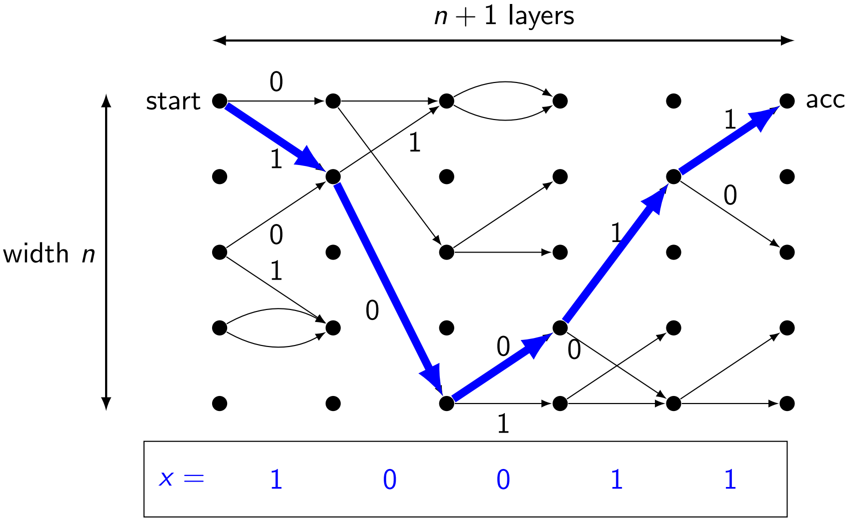
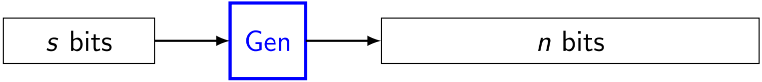


- Computes function $f : \{0,1\}^n \to \{0,1\}$

Fooling / Hitting ROBPs



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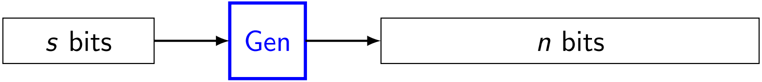

Pseudorandom generator: For every width-n ROBP,

$$\left|\Pr_x[f(x) = 1] - \Pr_z[f(\mathrm{Gen}(z)) = 1]\right| \leq \varepsilon$$

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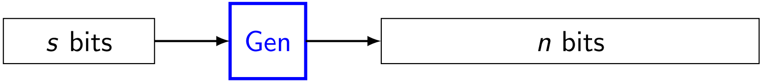

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Hitting Set Generator

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- Assume $\Pr[\text{accept}] = \varepsilon \ll 1/n^3$
- **Lemma**: There is a vertex u so that

$$\Pr[\text{reach } u] \geq \frac{1}{2n^3} \quad \text{and} \quad \Pr[\text{accept} \mid \text{reach } u] \geq \varepsilon \boldsymbol{n}.$$

Proof of lemma ($\exists u, \Pr[u] \geq \frac{1}{2n^3} \wedge \Pr[\text{acc} \mid u] \geq \varepsilon n$)

- Say u is a milestone if $\Pr[\text{accept} \mid \text{reach } u] \in [\varepsilon n, 2\varepsilon n]$

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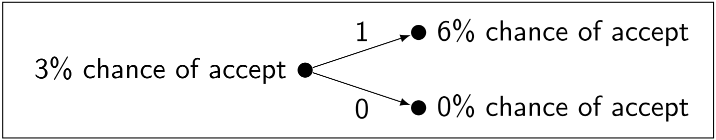

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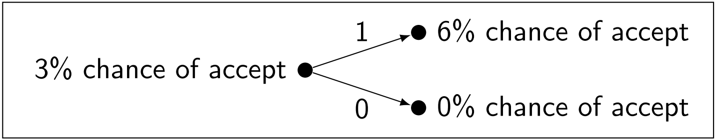

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$$
\begin{tikzcd}
 & \bullet \text{ 6\% chance of accept} \\
\text{3\% chance of accept } \bullet \arrow[ur, "1"] \arrow[dr, "0"'] & \\
 & \bullet \text{ 0\% chance of accept}
\end{tikzcd}
$$

- $$\varepsilon = \Pr[\text{accept}] \leq \sum_{u \text{ milestone}} \Pr[\text{reach } u \text{ and accept}]$$
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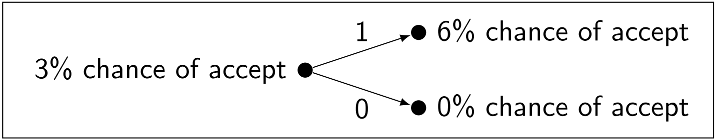

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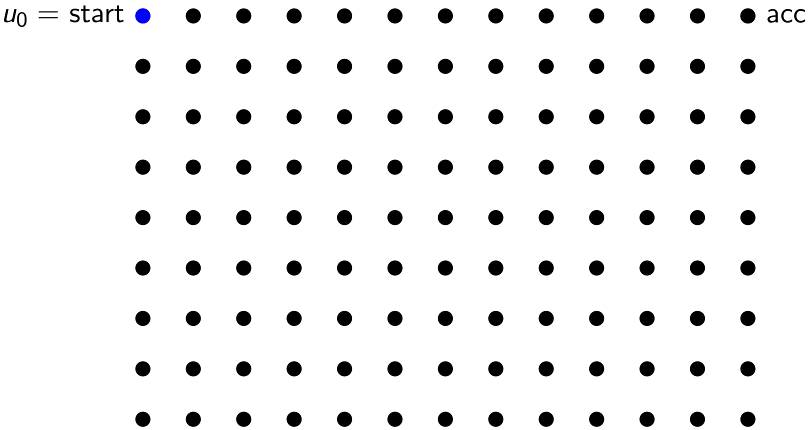
- $\#$ milestones $\leq n^2$, so for some milestone u, $\Pr[\text{reach } u] \geq \frac{1}{2n^3}$ $\square$

Iterating the structural lemma



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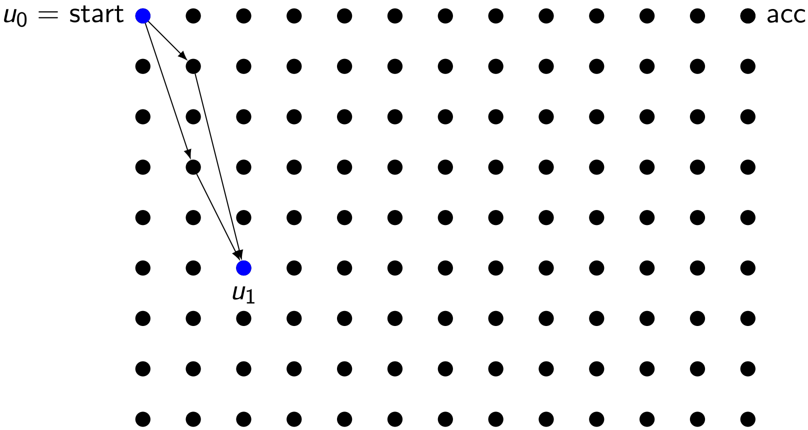

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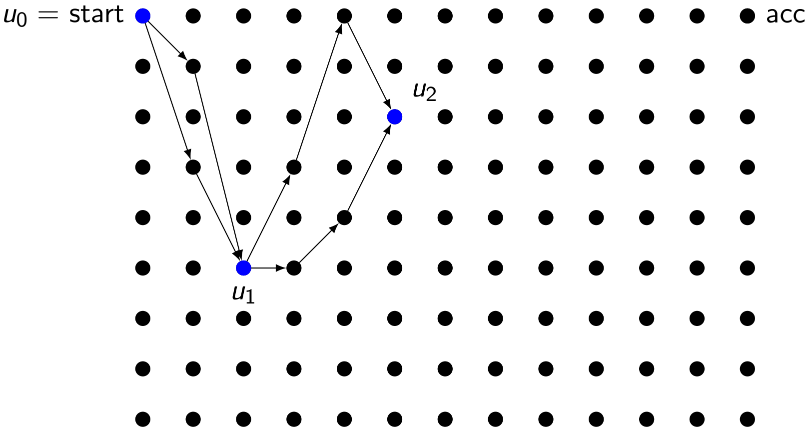

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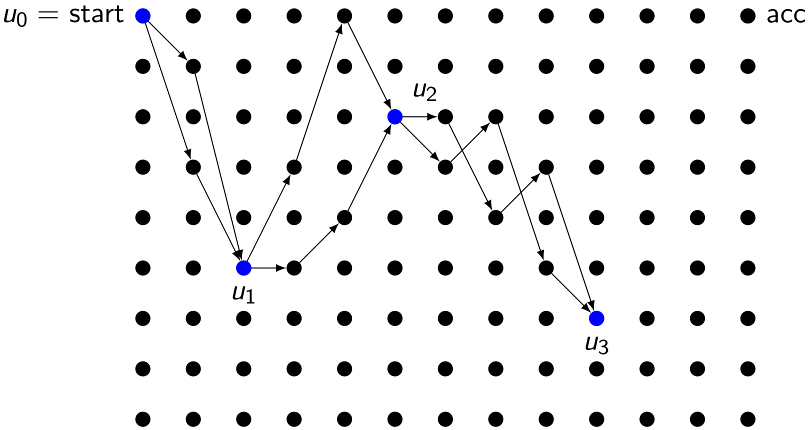

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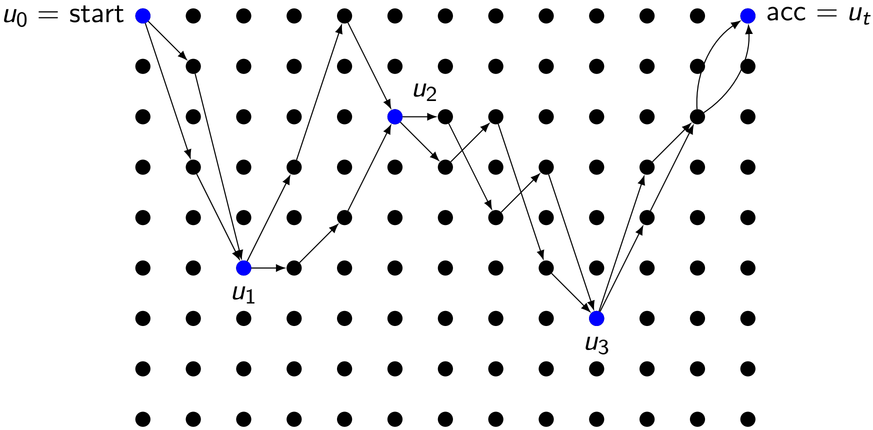

$\Pr[\text{accept}] = \varepsilon$ $\quad n\varepsilon \quad n^2\varepsilon \quad n^3\varepsilon \quad n^t\varepsilon = 1$

Idea of our HSG

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- Use a "hitter" to recycle the seed of Nisan's generator from one hop to the next

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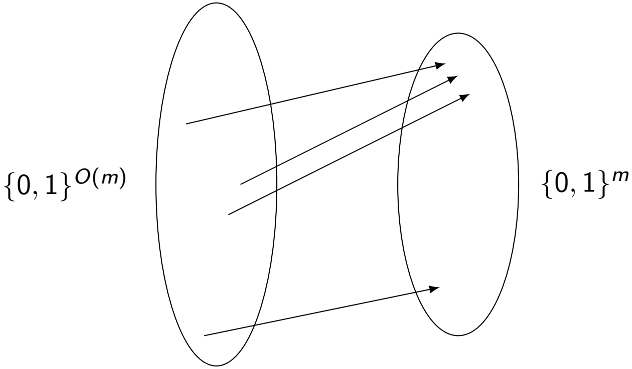

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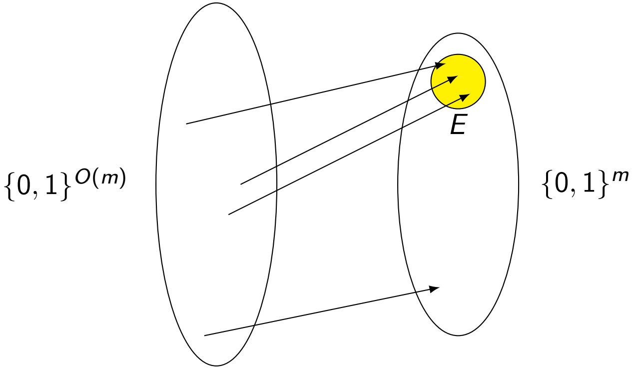

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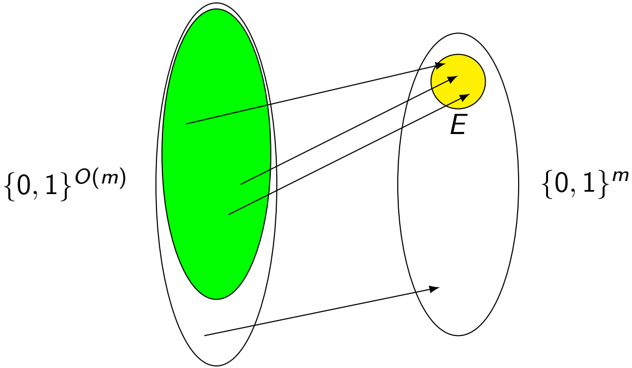
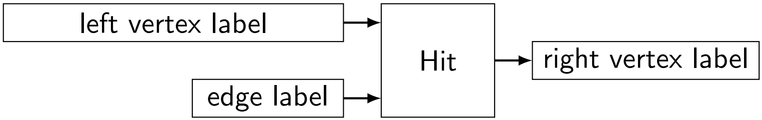


Hitter as a function



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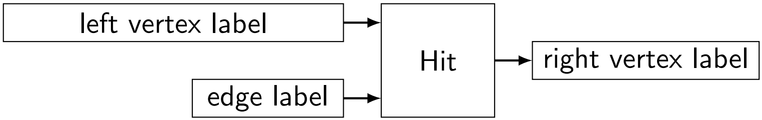

- For any E with density$(E) \geq \theta$,

$$\Pr_x[\exists y, \mathsf{Hit}(x, y) \in E] \geq 1 - \delta$$

Our HSG

$\boxed{x}$

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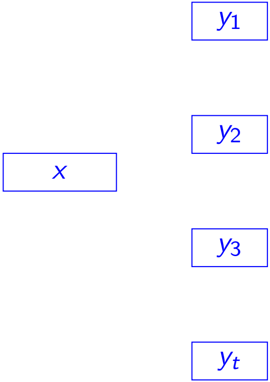

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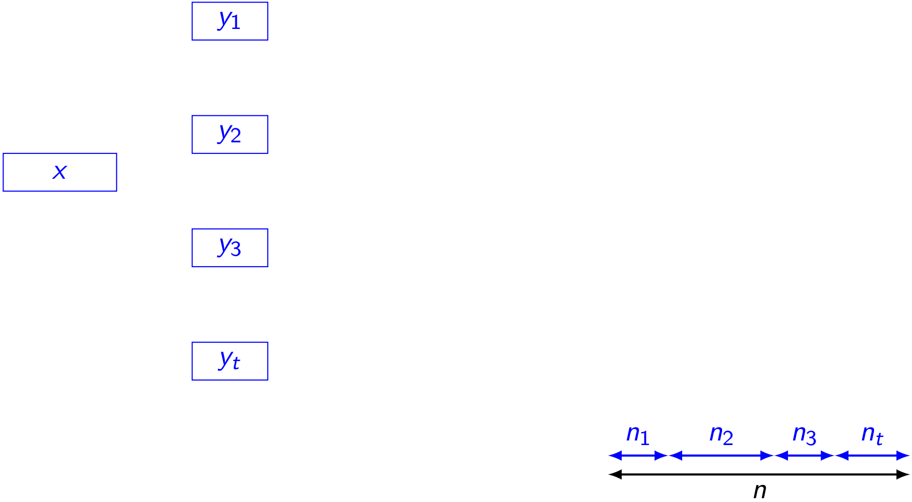

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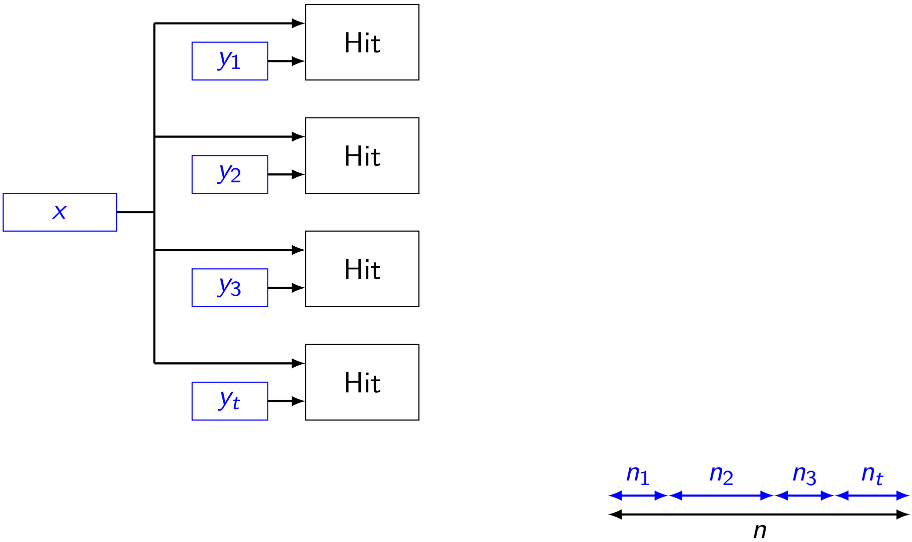

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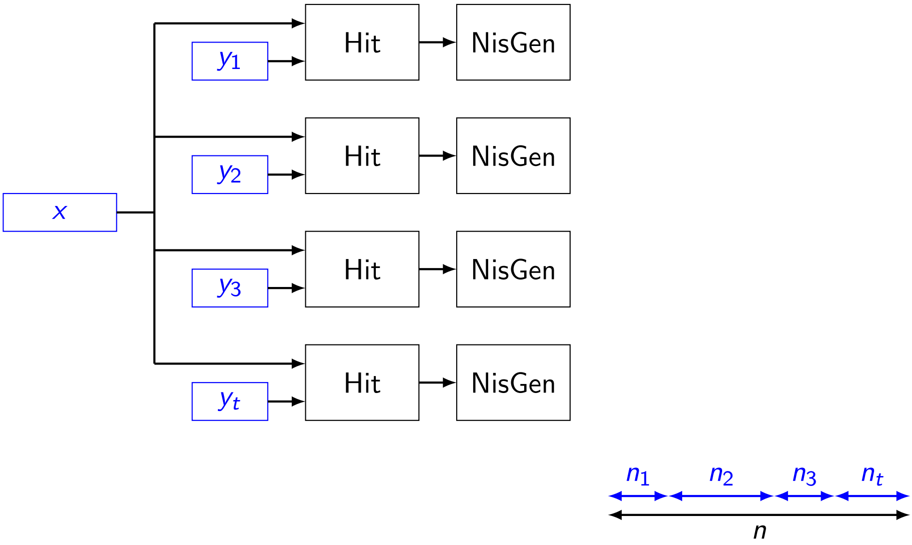

x
y_1
y_2
y_3
y_t
Hit
Hit
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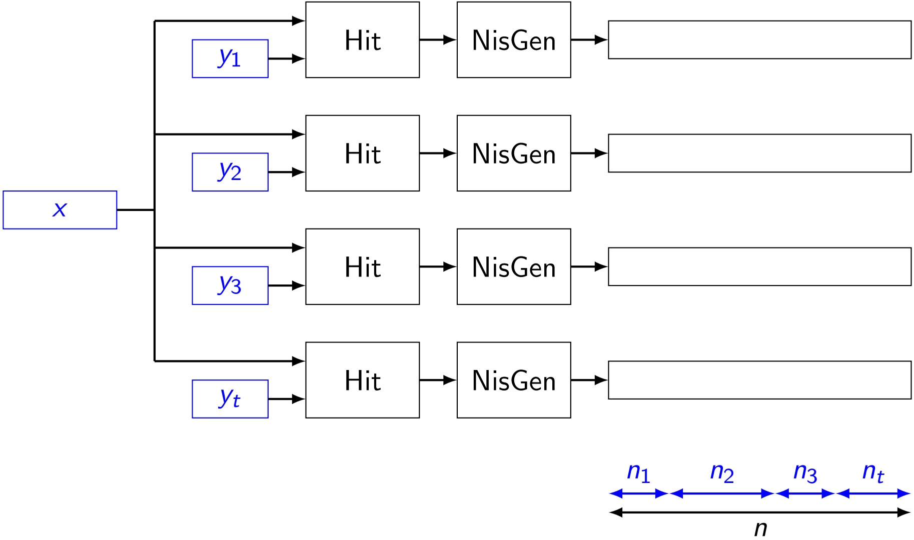

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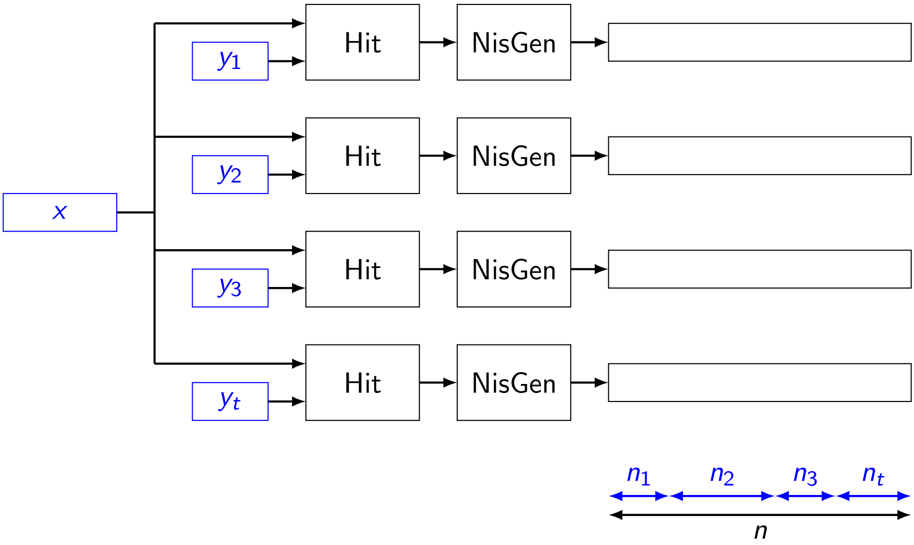
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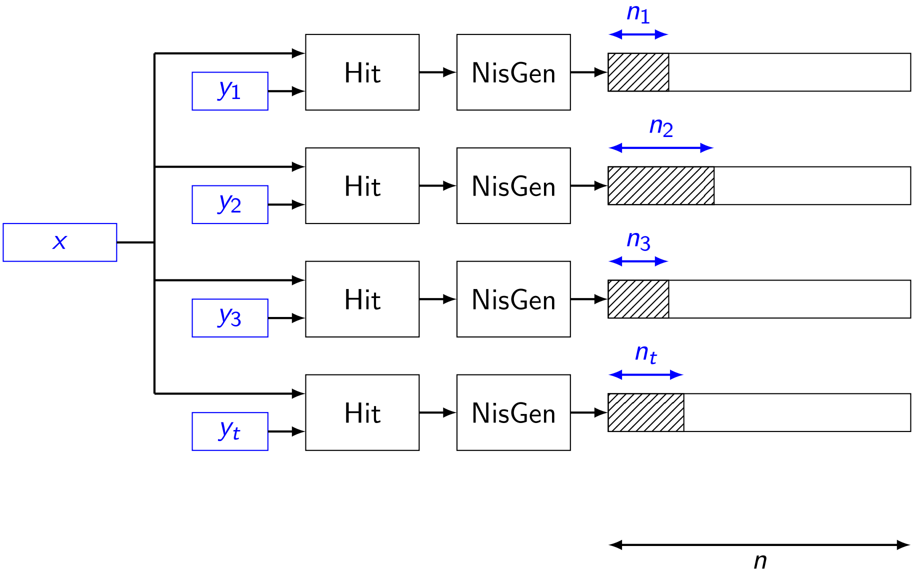
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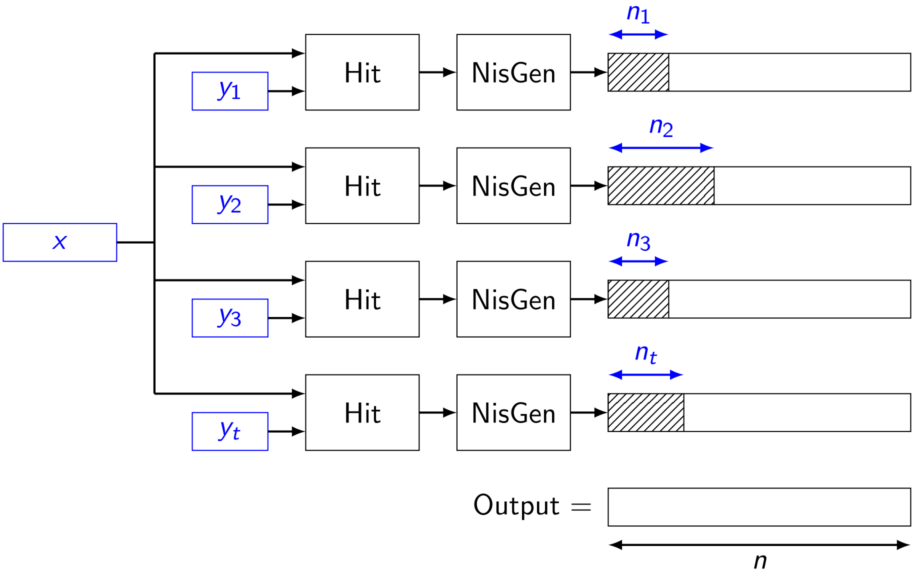

Our HSG in symbols

- For numbers $n_1, \ldots, n_t$ with $n_1 + \cdots + n_t = n$:

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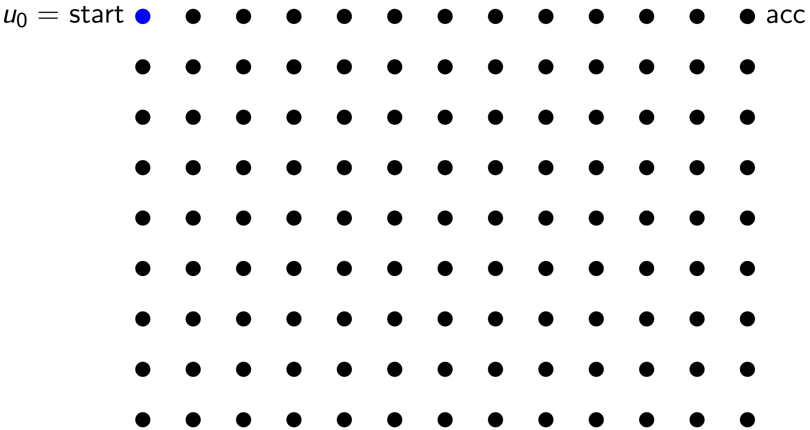

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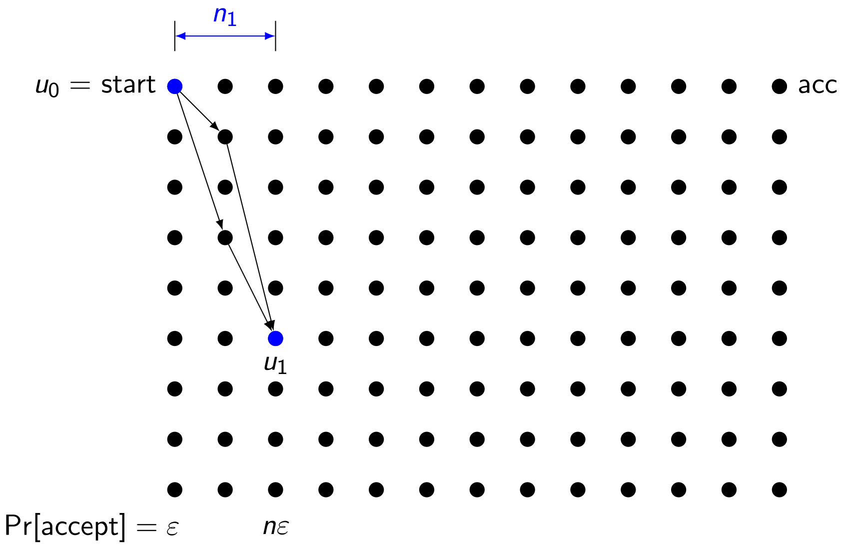

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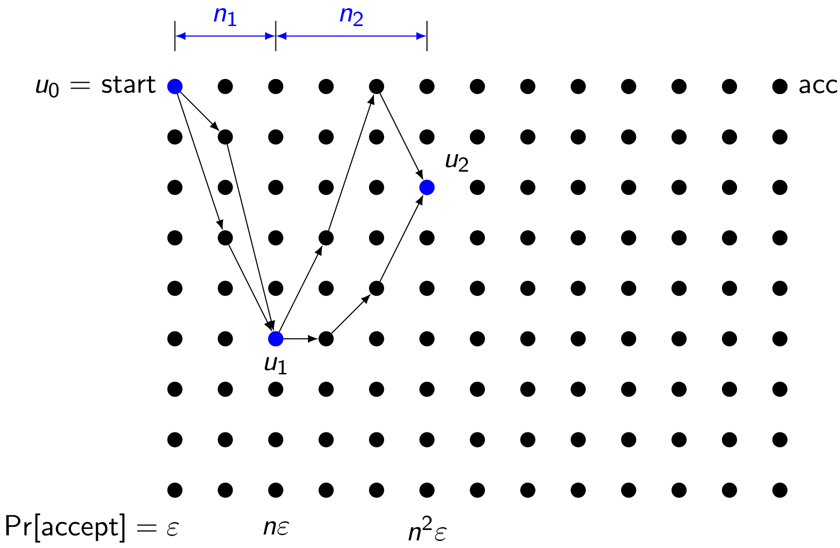

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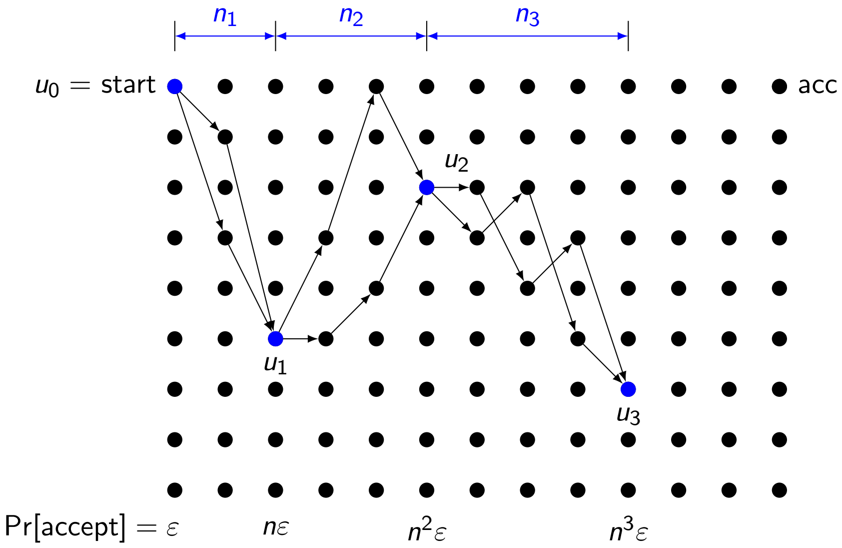

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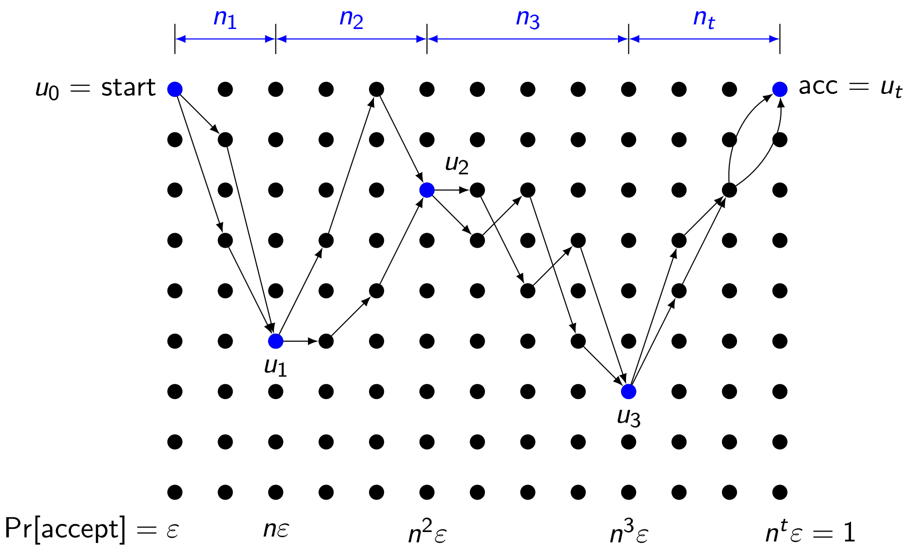

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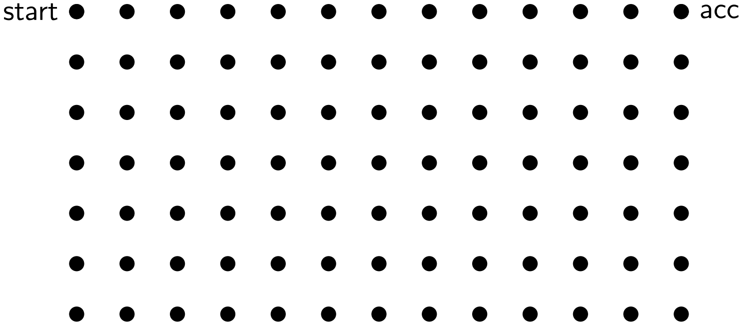

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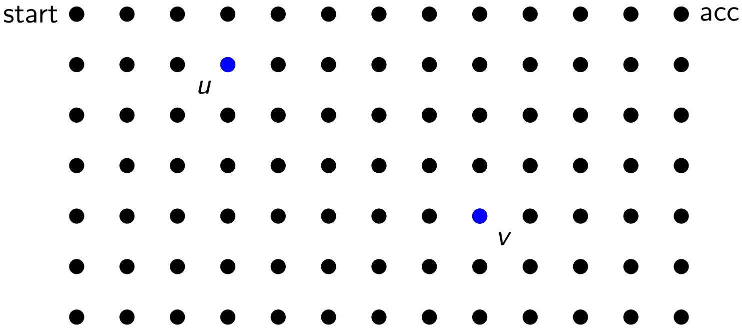

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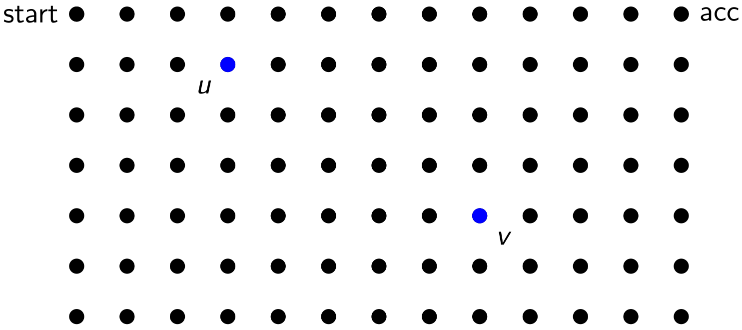

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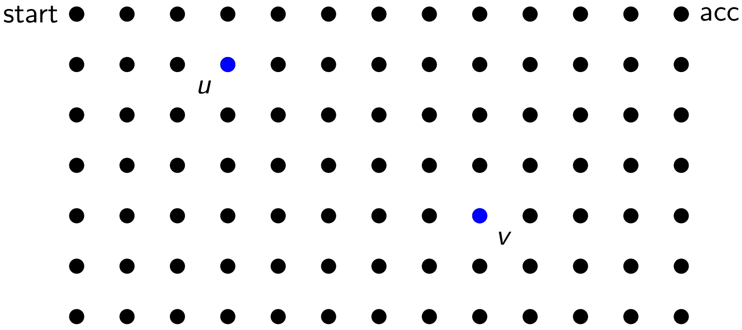

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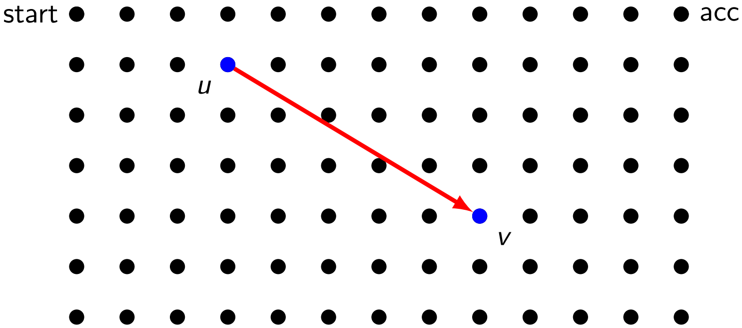

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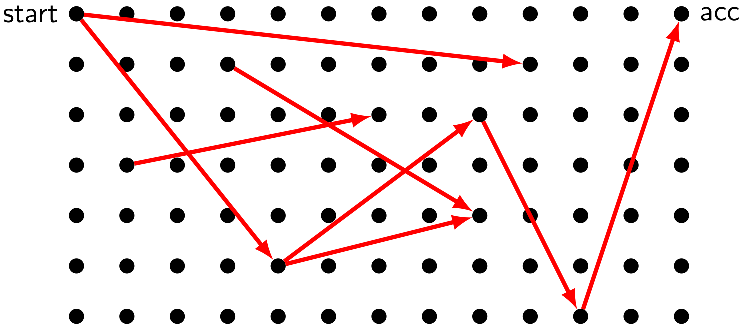

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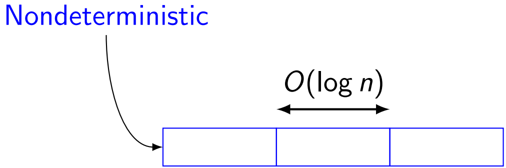

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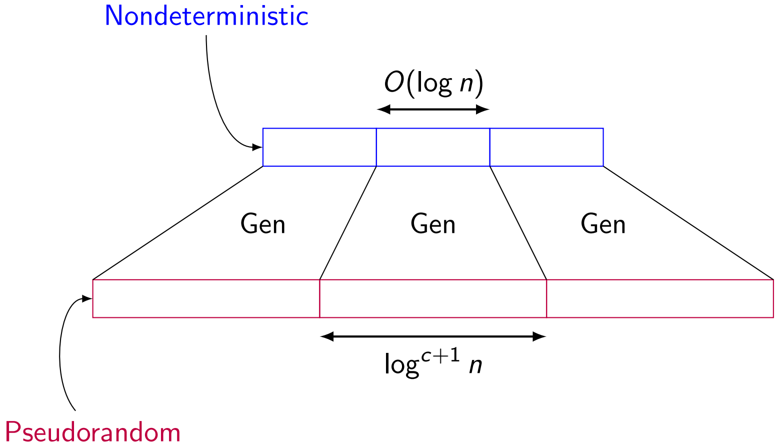

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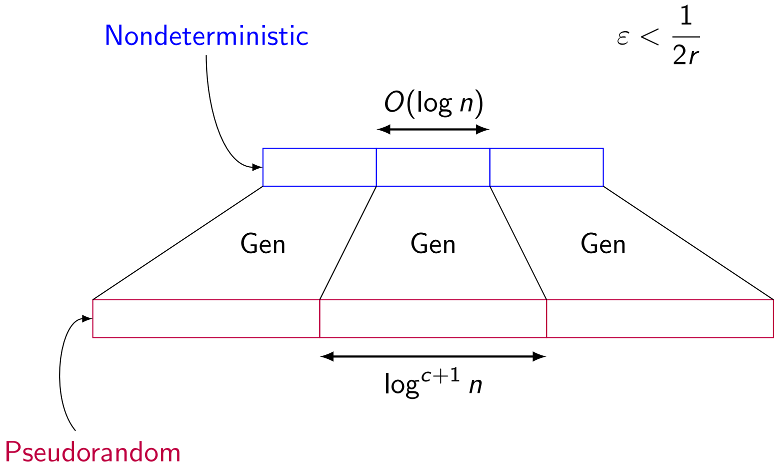

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The case polylog $n \ll r \ll n$

- **Theorem**: HSG for width-n, length-r ROBPs with seed length

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- **Proof**: Plug in PRG of [Armoni '98]

Open questions

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- ACR '96: Explicit HSG for circuits $\implies$ $\mathbf{P} = \mathbf{BPP}$. Similar theorem for **BPL**?

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$$(\textbf{BPL} \text{ with } r \text{ coins}) = \left(\textbf{BPL} \text{ with } \frac{r}{\log^c n} \text{ coins}\right)$$

  - True for $r \leq 2^{\log^{0.99} n}$ by Nisan-Zuckerman

- ACR '96: Explicit HSG for circuits $\implies$ $\mathbf{P} = \mathbf{BPP}$. Similar theorem for **BPL**?

- Thanks! Questions?