Simple Optimal Hitting Sets for Small-Success **RL**

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[2] Supported by NSF Grant CCF-1526952, NSF Grant CCF-1705028, and a Simons Investigator Award (#409864)

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- $L \in \mathbf{RL}$ if there is a randomized log-space algorithm A that always halts such that

$$x \in L \implies \Pr[A(x) \text{ accepts}] \geq 1/2$$
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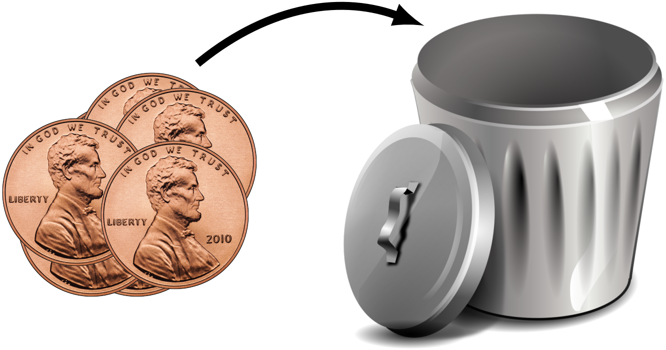

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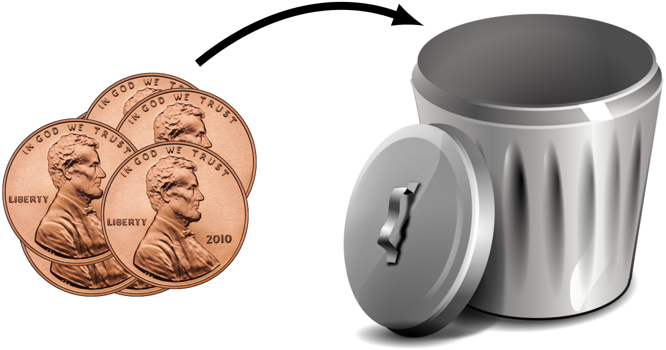
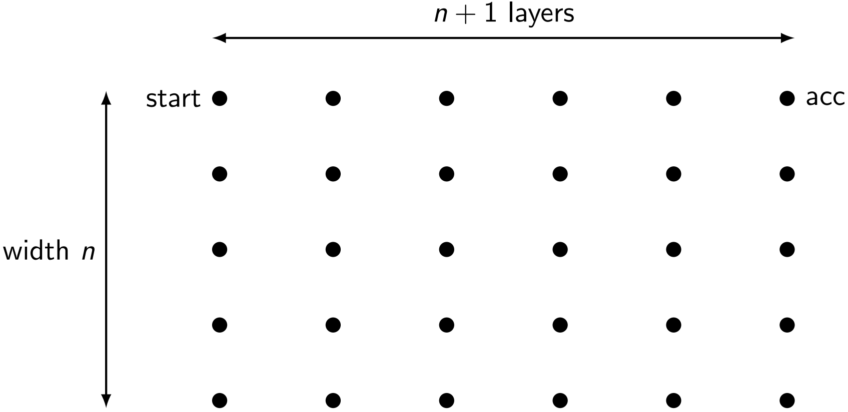


Read-once branching programs



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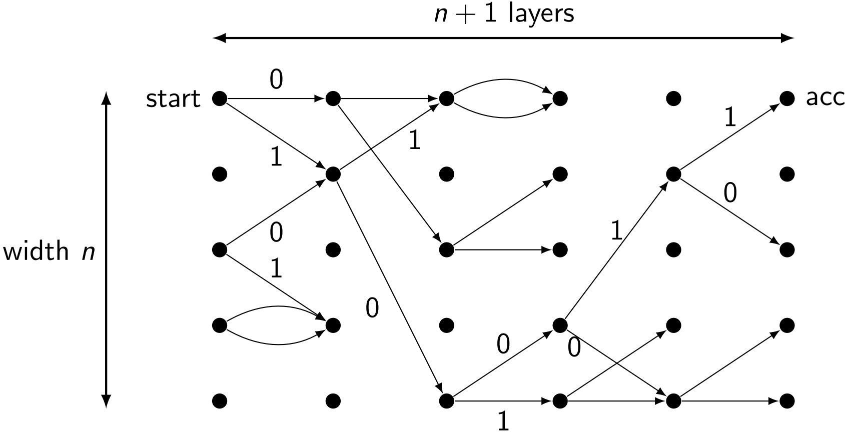

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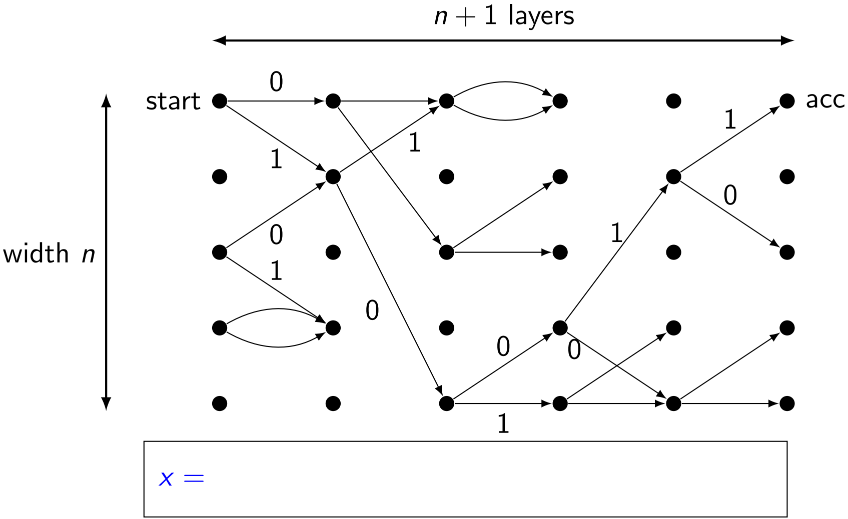

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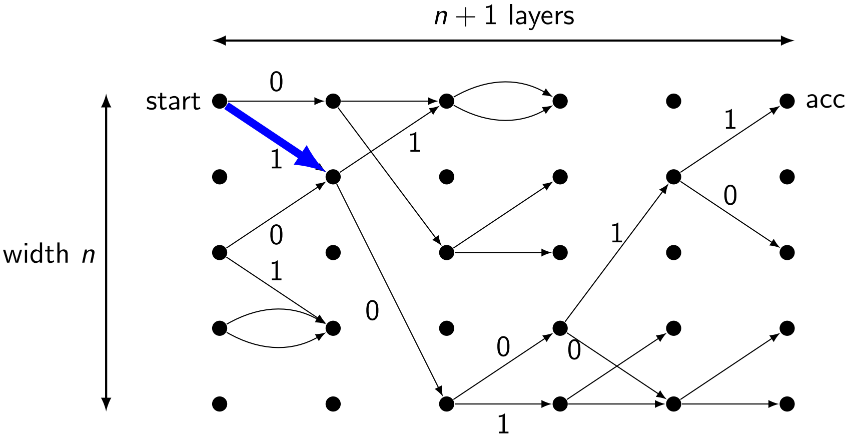

$x = \quad 1$

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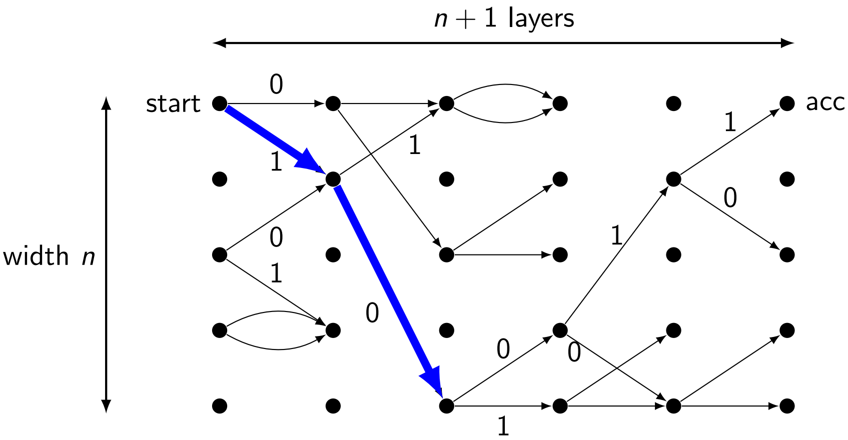

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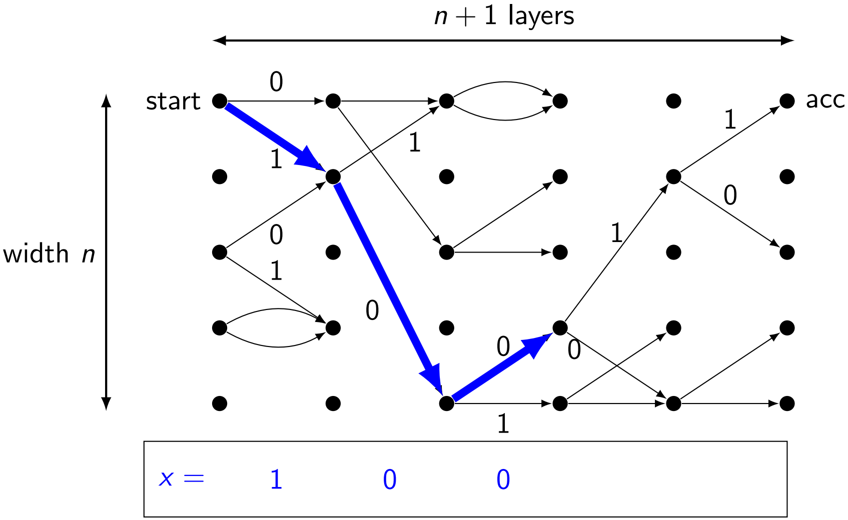

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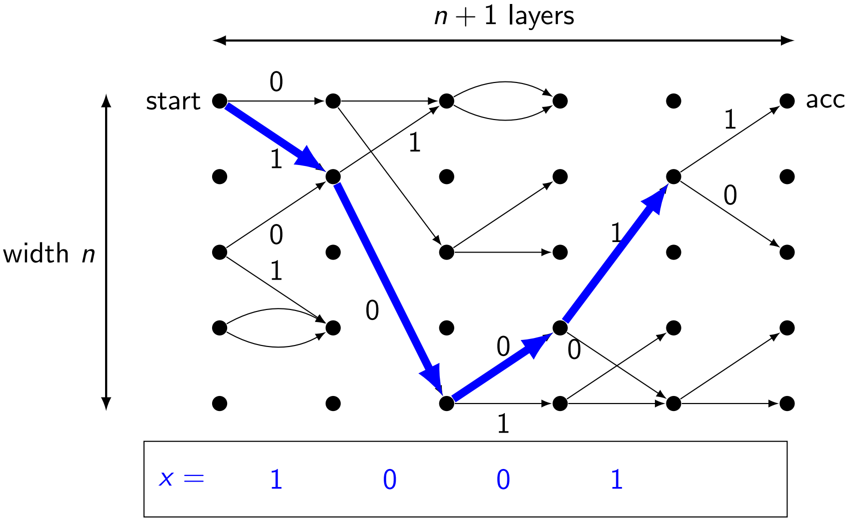

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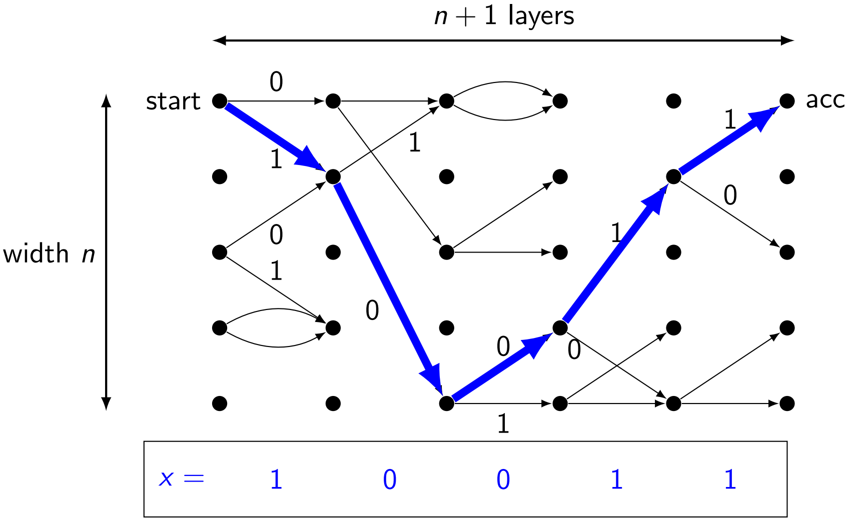

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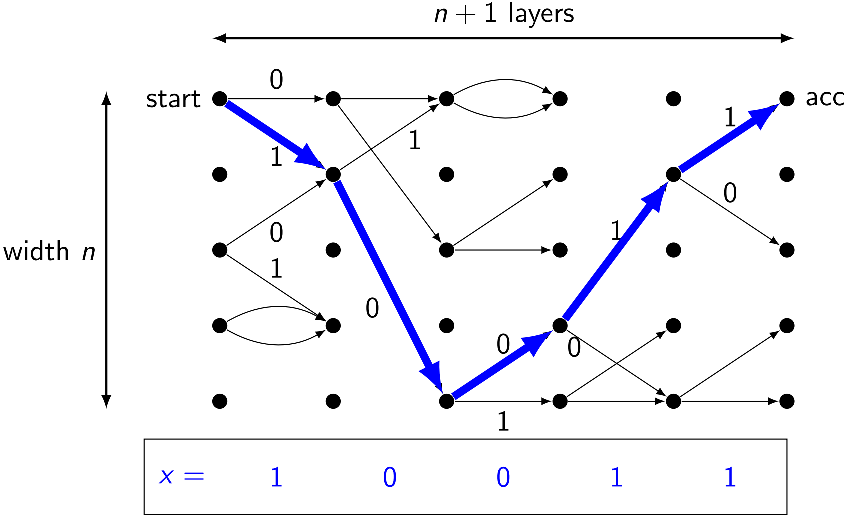
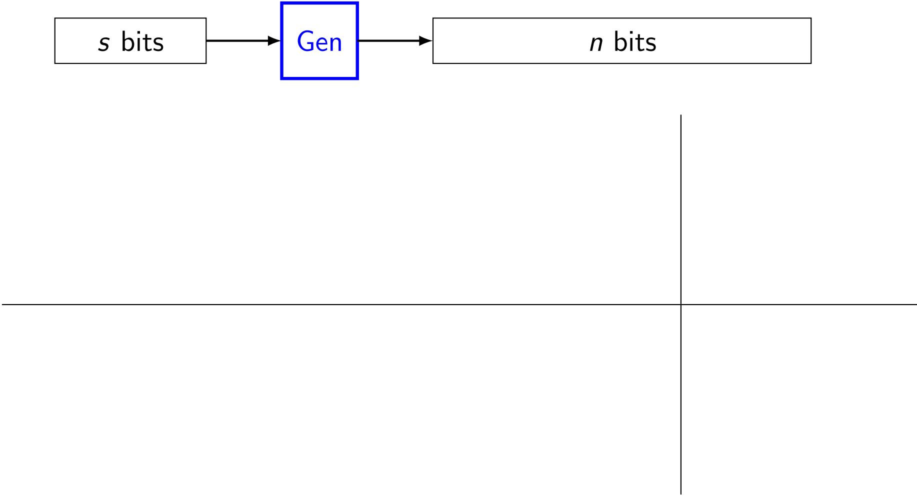


- Computes function $f : \{0,1\}^n \to \{0,1\}$

Fooling / Hitting ROBPs



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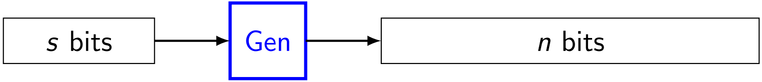

Pseudorandom generator: For every width-n ROBP,

$$|\Pr_x[f(x) = 1] - \Pr_z[f(\mathrm{Gen}(z)) = 1]| \leq \varepsilon$$

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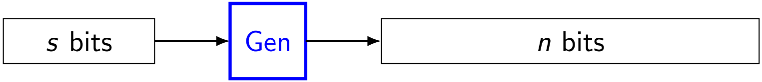

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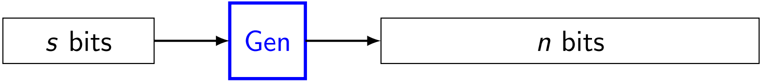

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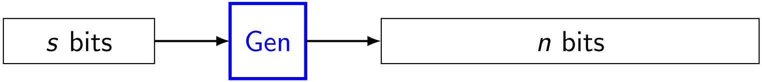

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- **This work**: HSG with seed length

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Comparison with [BCG '18]

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Hitting Set Generator

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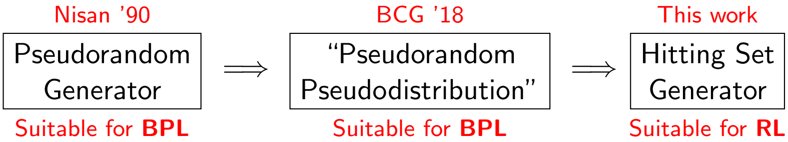

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- Assume $\Pr[\text{accept}] = \varepsilon \ll 1/n^3$
- **Lemma**: There is a vertex u so that

$$\Pr[\text{reach } u] \geq \frac{1}{2n^3} \quad \text{and} \quad \Pr[\text{accept} \mid \text{reach } u] \geq \varepsilon \boldsymbol{n}.$$

Proof of lemma ($\exists u, \Pr[u] \geq \frac{1}{2n^3} \wedge \Pr[\text{acc} \mid u] \geq \varepsilon n$)

- Say u is a milestone if $\Pr[\text{accept} \mid \text{reach } u] \in [\varepsilon n, 2\varepsilon n]$

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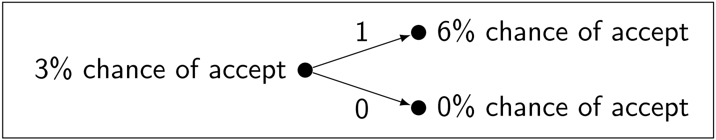

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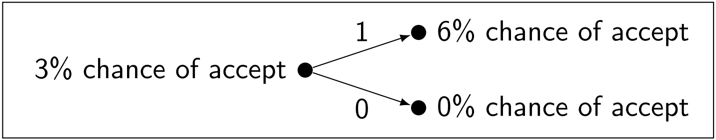

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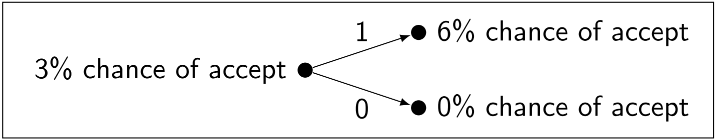

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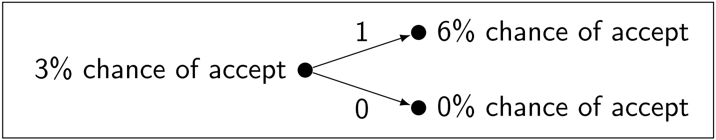

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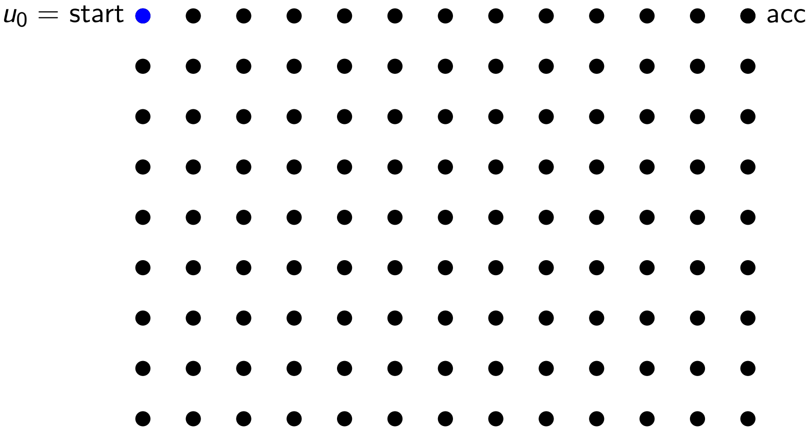
- $\#$ milestones $\leq n^2$, so for some milestone u, $\Pr[\text{reach } u] \geq \frac{1}{2n^3}$ $\square$

Iterating the structural lemma



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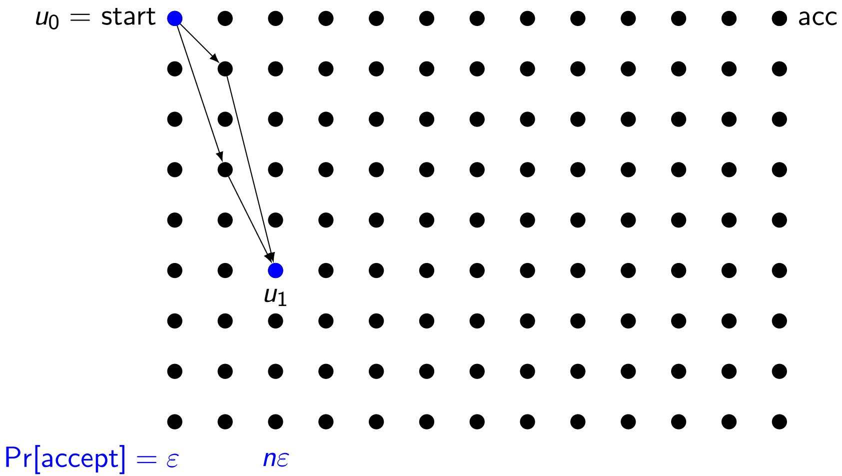

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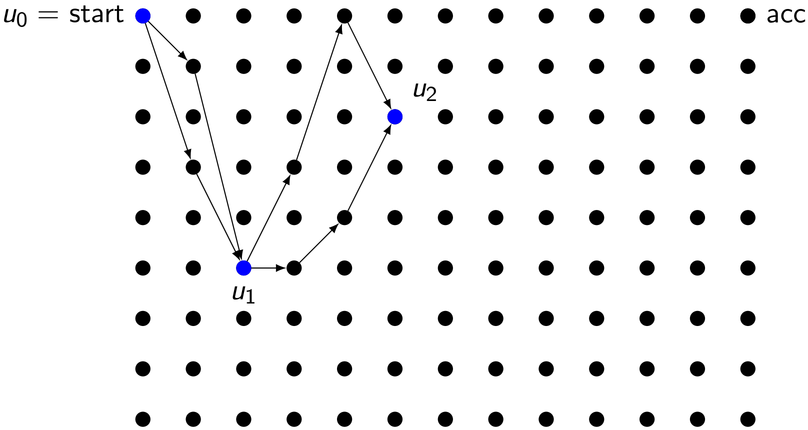

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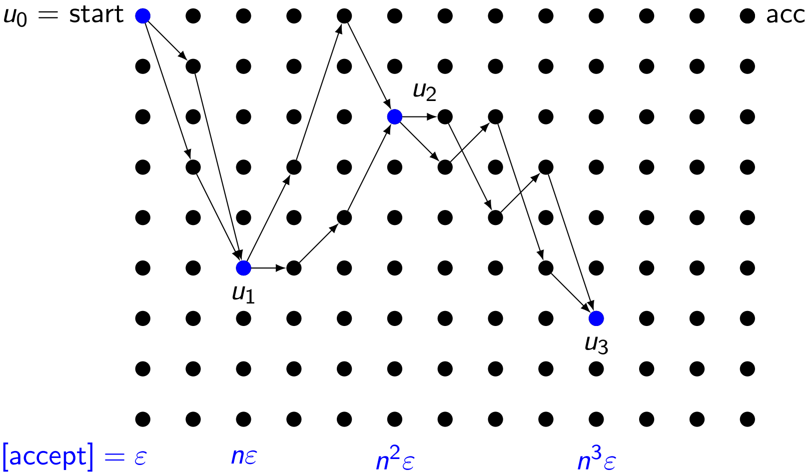

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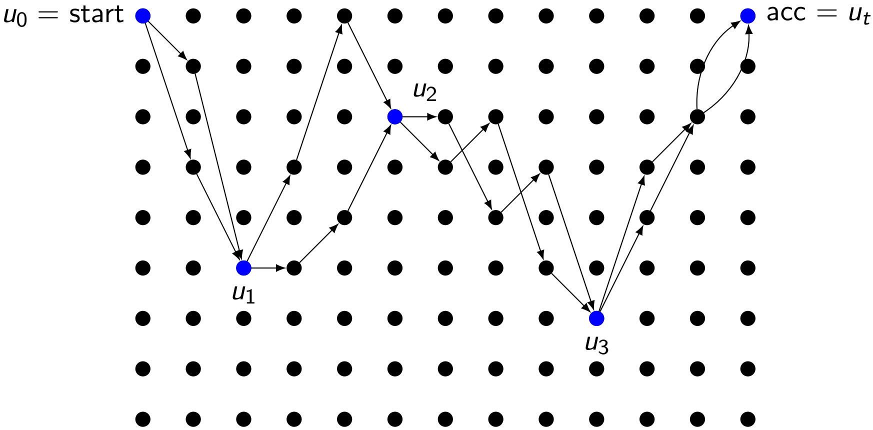

Pr[accept] $= \varepsilon$ $\quad n\varepsilon \quad n^2\varepsilon \quad n^3\varepsilon \quad n^t\varepsilon = 1$

Idea of our HSG

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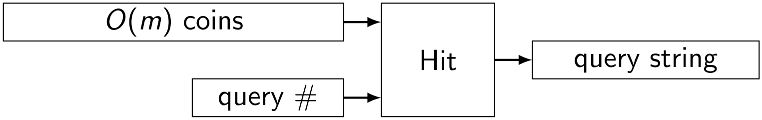

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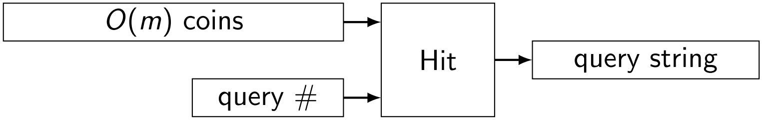


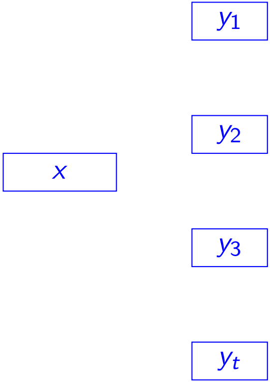
- For any E with density$(E) \geq \theta$,

$$\Pr_x[\exists y, \mathrm{Hit}(x,y) \in E] \geq 1 - \delta$$

Our HSG

x

Our HSG



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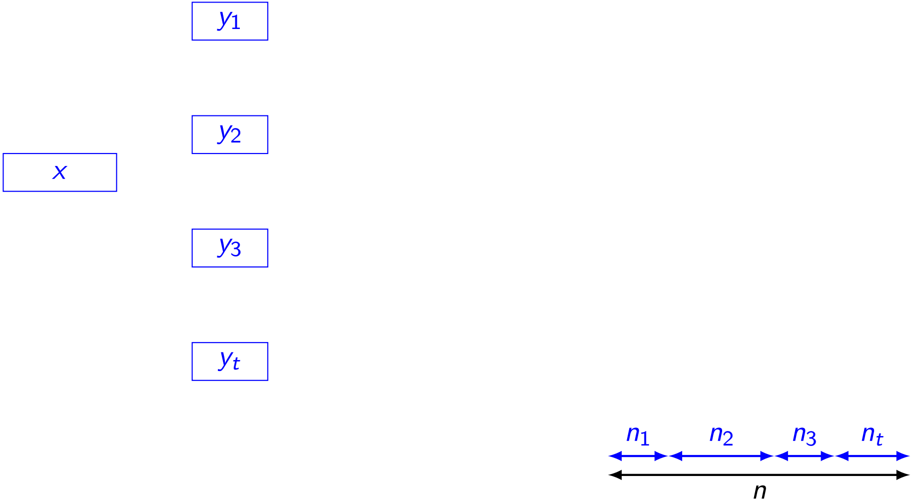

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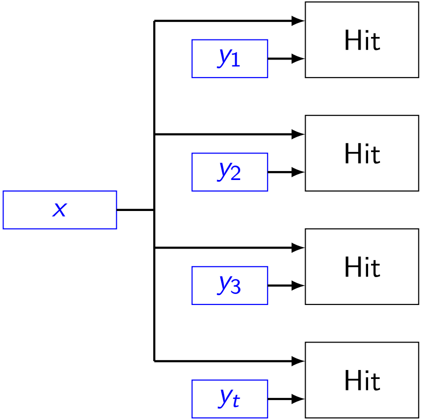

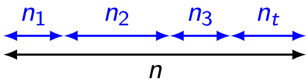

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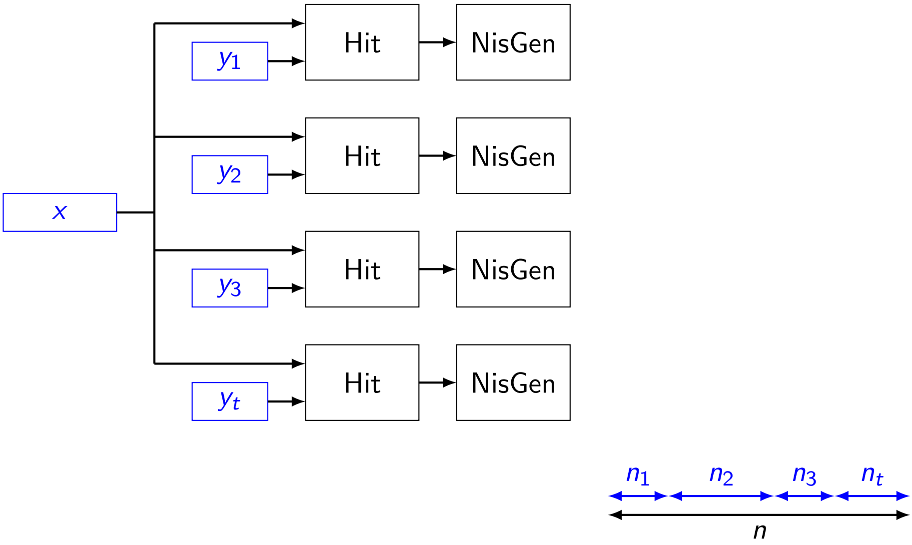

Hit
NisGen
y_1
Hit
NisGen
y_2
x
Hit
NisGen
y_3
Hit
NisGen
y_t
n_1
n_2
n_3
n_t
n

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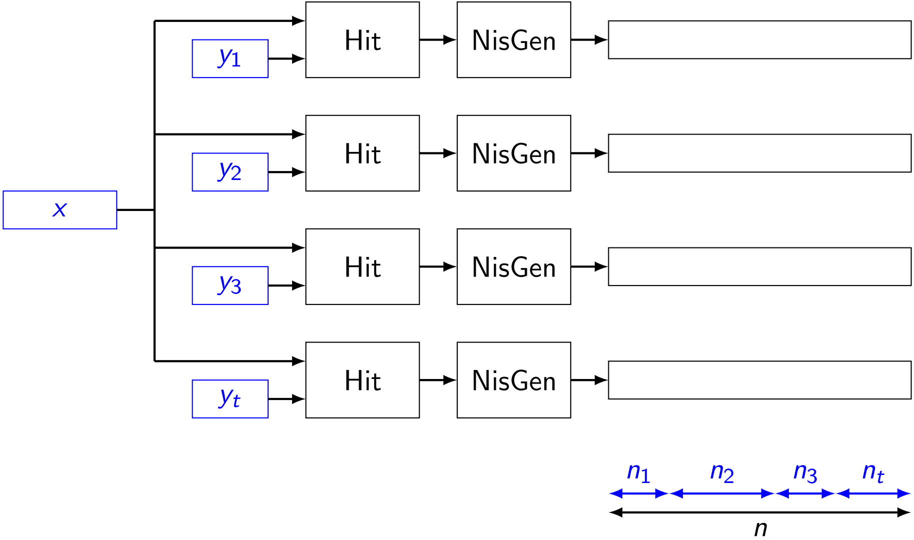

y_1
Hit
NisGen
y_2
Hit
NisGen
x
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Hit
NisGen
y_t
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NisGen
n_1
n_2
n_3
n_t
n

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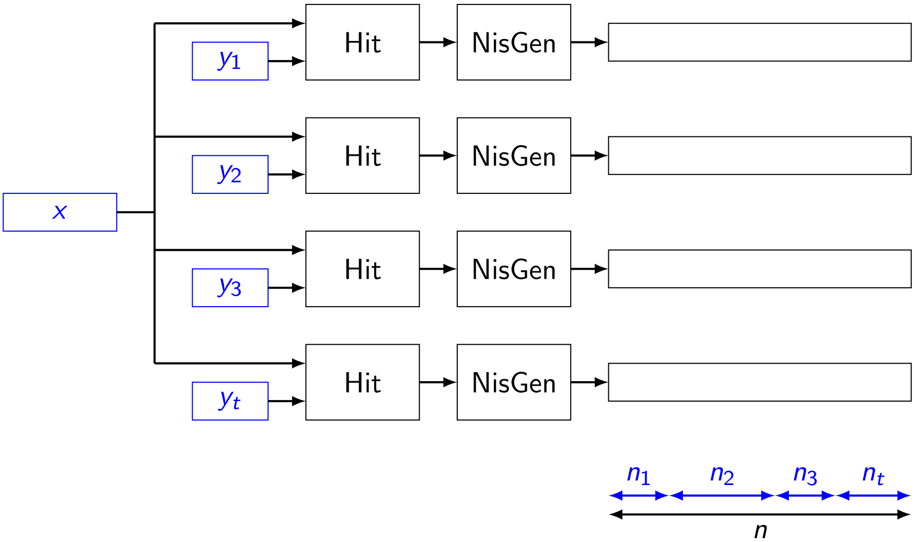

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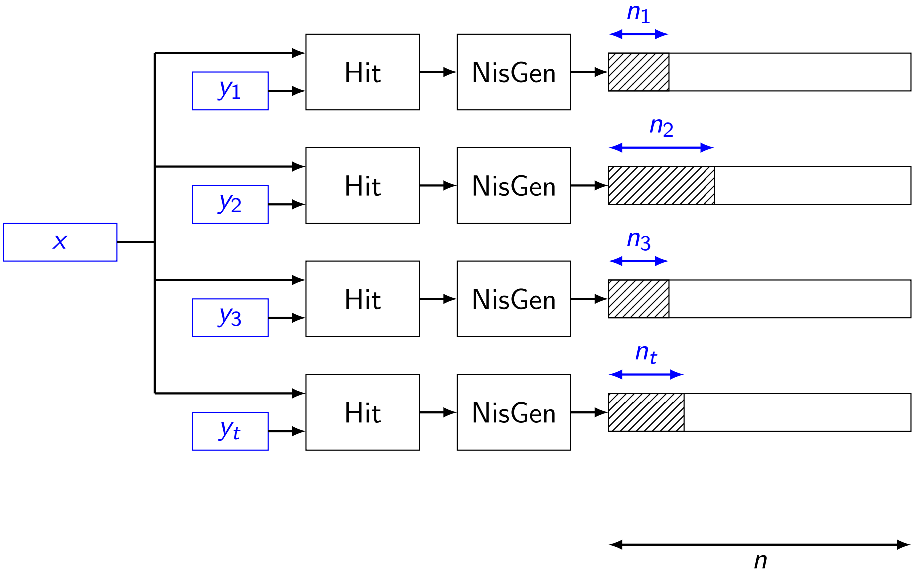

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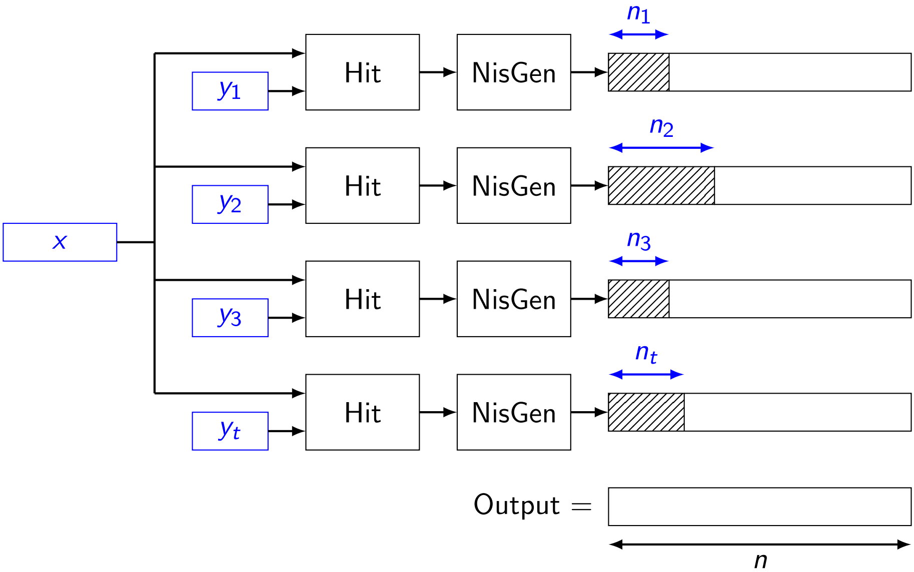
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Hit
Hit
Hit
Hit
NisGen
NisGen
NisGen
NisGen
Output =
n

Our HSG in symbols

- For numbers $n_1, \ldots, n_t$ with $n_1 + \cdots + n_t = n$:

$$\begin{aligned} &\mathrm{Gen}(x, y_1, \ldots, y_t, n_1, \ldots, n_t) = \\ &\quad \mathrm{NisGen}(\mathrm{Hit}(x, y_1))|_{n_1} \circ \cdots \circ \mathrm{NisGen}(\mathrm{Hit}(x, y_t))|_{n_t} \in \{0,1\}^n \end{aligned}$$

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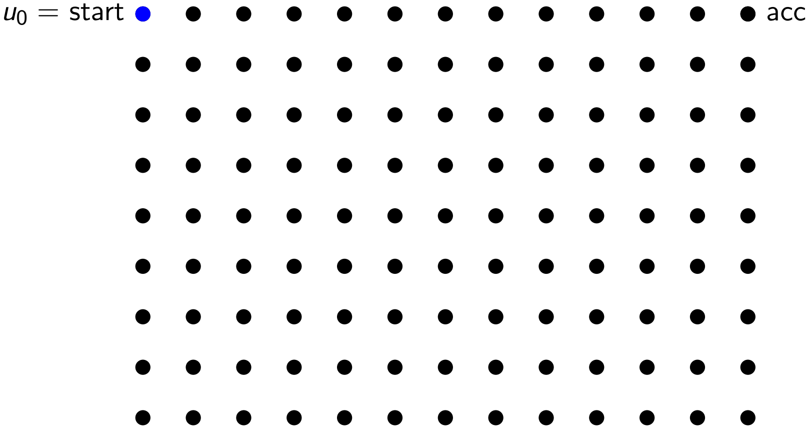
- So seed length $= O(\log^2 n + \log(1/\varepsilon))$

Proof of correctness of our HSG



$\Pr[\text{accept}] = \varepsilon$

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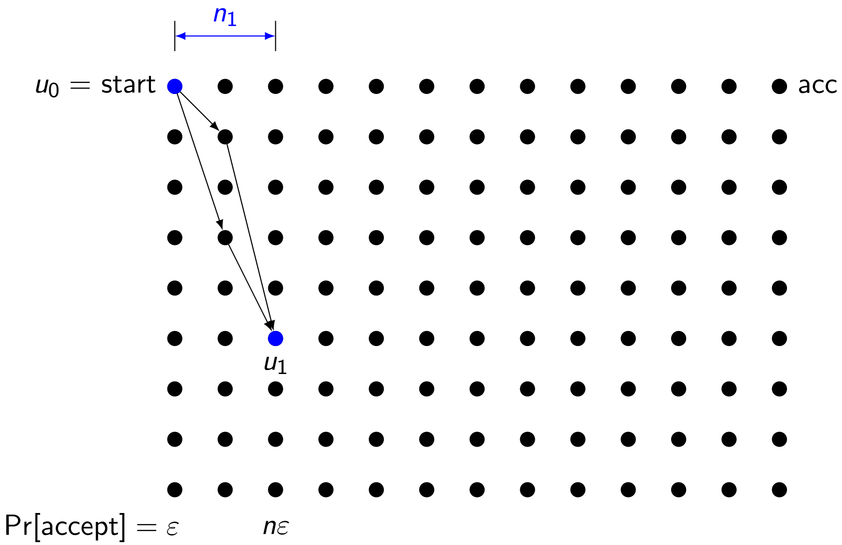

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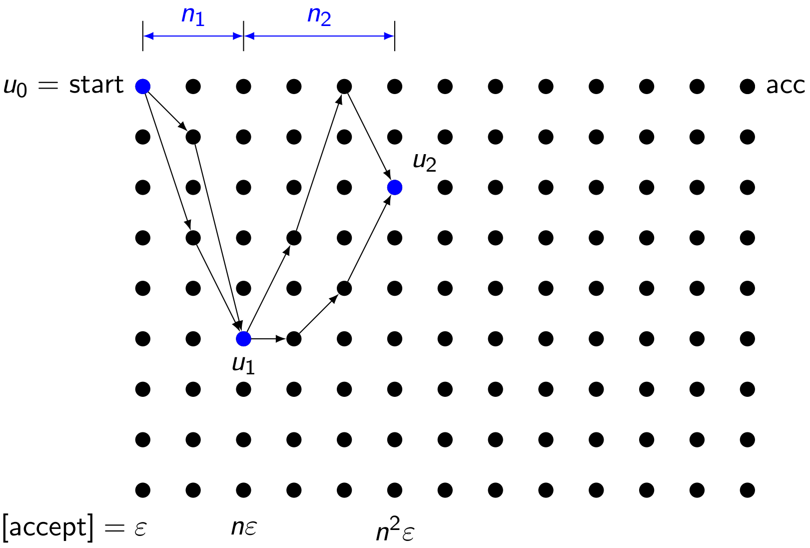

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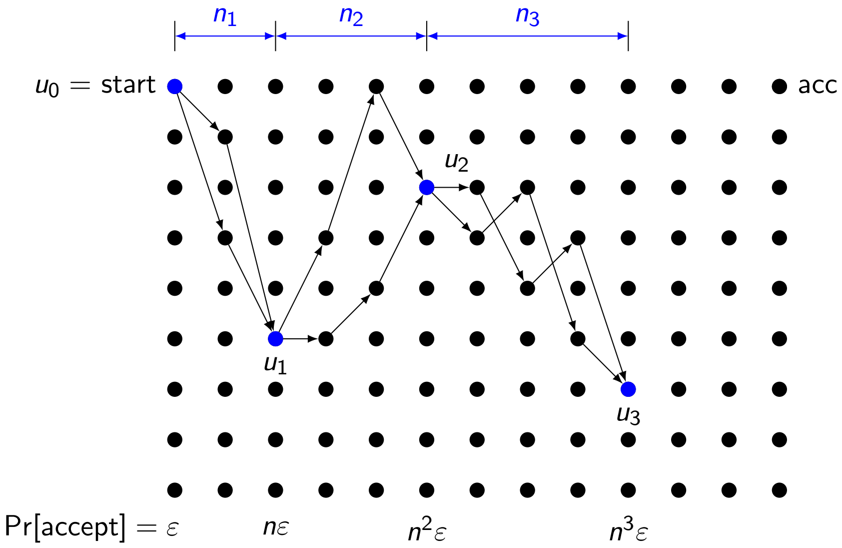

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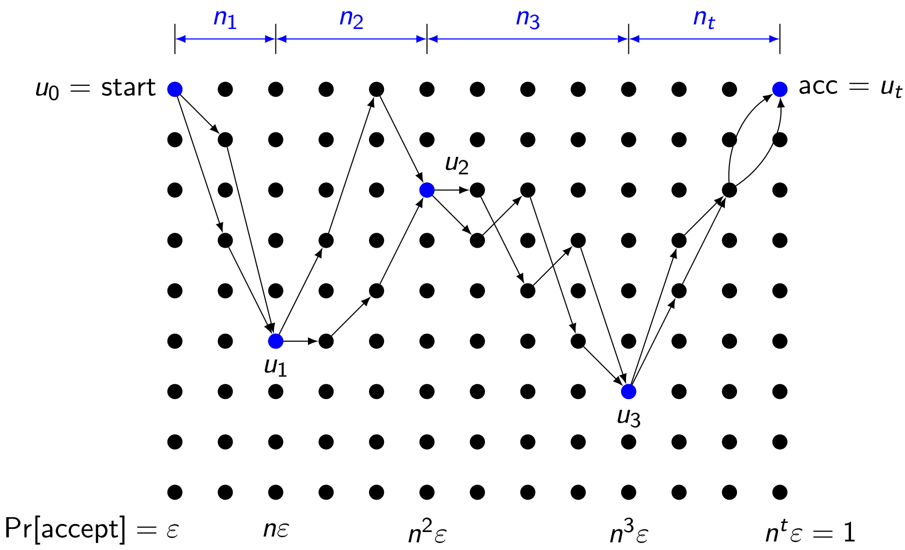

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- Define $E_i \subseteq \{0,1\}^m$ by

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Additional results

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$$(\varepsilon\text{-success } \mathbf{RL}) \subseteq \mathbf{DSPACE}(\log^{3/2} n + \log n \log\log(1/\varepsilon))$$

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- **Theorem**: For any $r = r(n)$, for any constant c,

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Open questions

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