Targeted Pseudorandom Generators, Simulation Advice Generators, and Derandomizing Logspace

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  - Assume the following derandomization statement:

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L vs. BPL

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- Best derandomization (Saks, Zhou '98):

$$\mathrm{BPL} \subseteq \mathrm{DSPACE}(\log^{3/2} n)$$

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$$\mathsf{BPL} \subseteq \bigcap_{\alpha > 0} \mathsf{DSPACE}(\log^{1+\alpha} n).$$
- Equivalence of PRGs and derandomization would itself give a derandomization!

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Outline

- ✓ Simplified statement of main result
- Proof sketch of main result
  - Saks-Zhou theorem, revisited
- Proof sketch of Saks-Zhou-Armoni theorem
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  - Targeted PRGs
  - Simulation advice generators

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- Nonuniform model of $\log n$ space: n-state automaton
- $Q^m(q; y) =$ final state if Q starts in state q, reads $y \in \{0,1\}^m$
- *Simulator* for automata: algorithm $\mathrm{Sim}(Q, q, x)$ such that

$$\mathrm{Sim}(Q, q, U_s) \sim_{\varepsilon} Q^m(q; U_m)$$

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- Crucial feature: Gen doesn't see "source code" (Q, q)!

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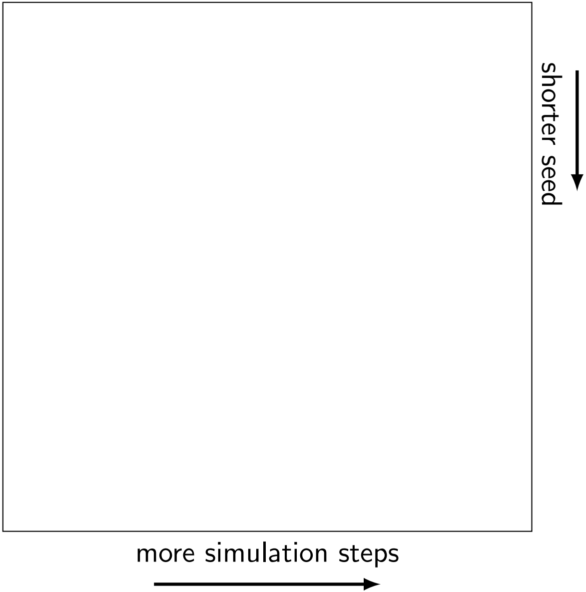
$$O(\log^{1.1} n + \log^{1.1} n) = O(\log^{1.1} n)$$

Proof of main result



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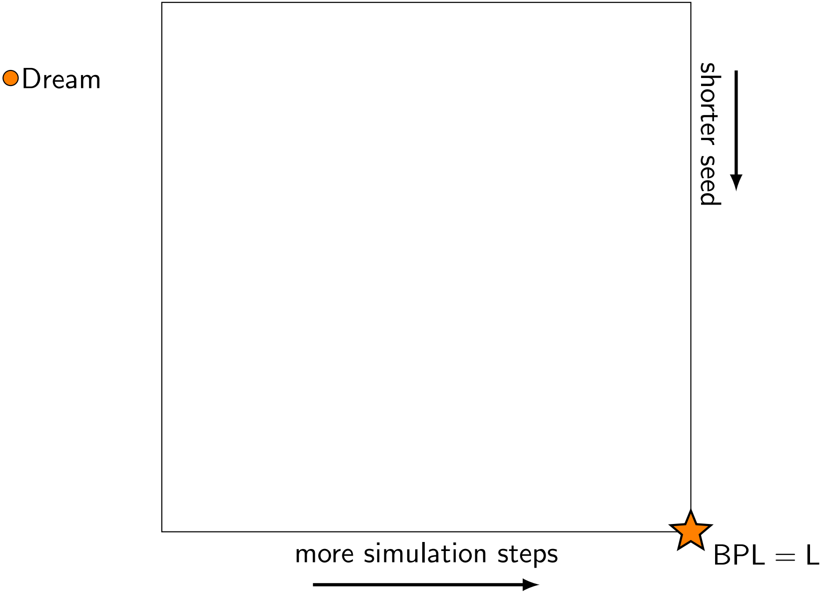

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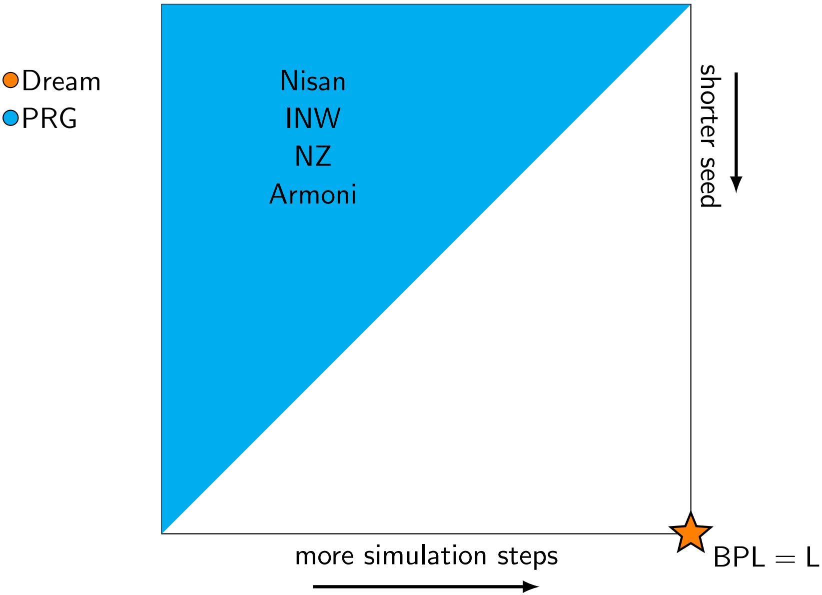

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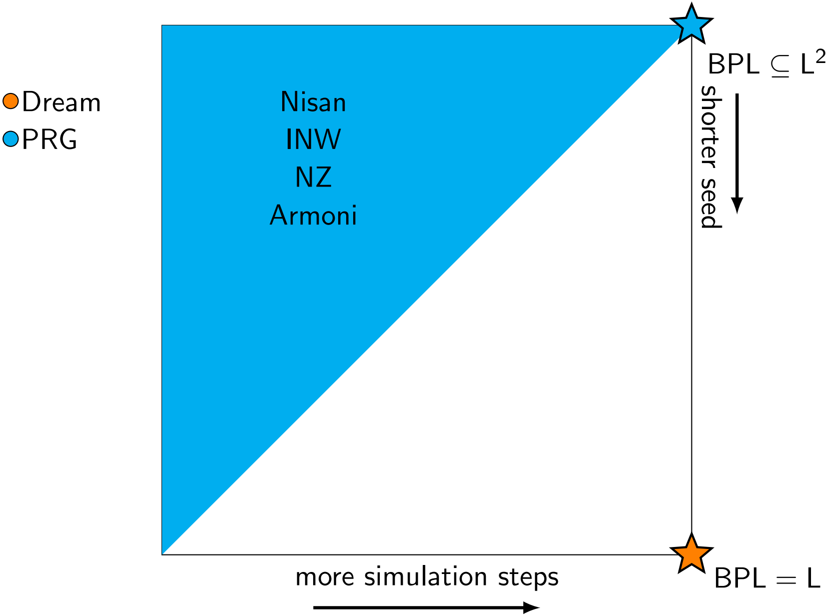

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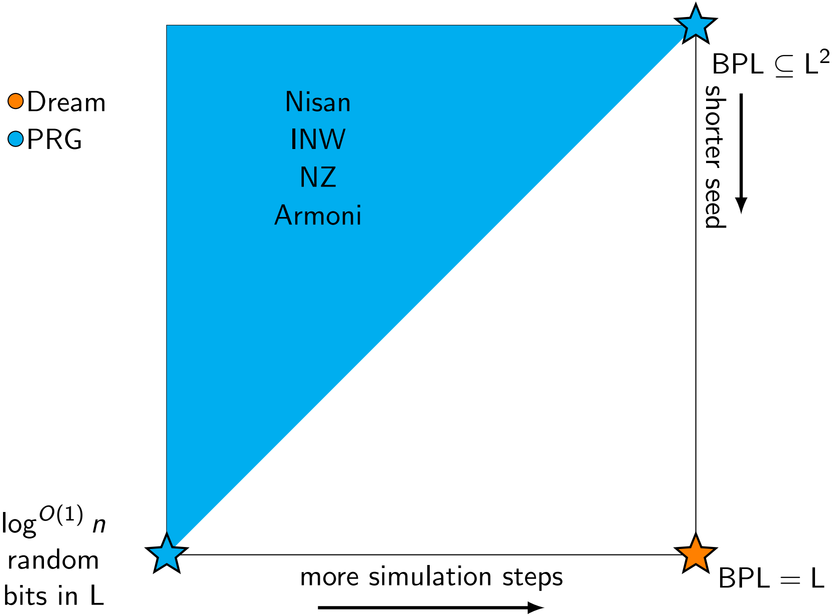

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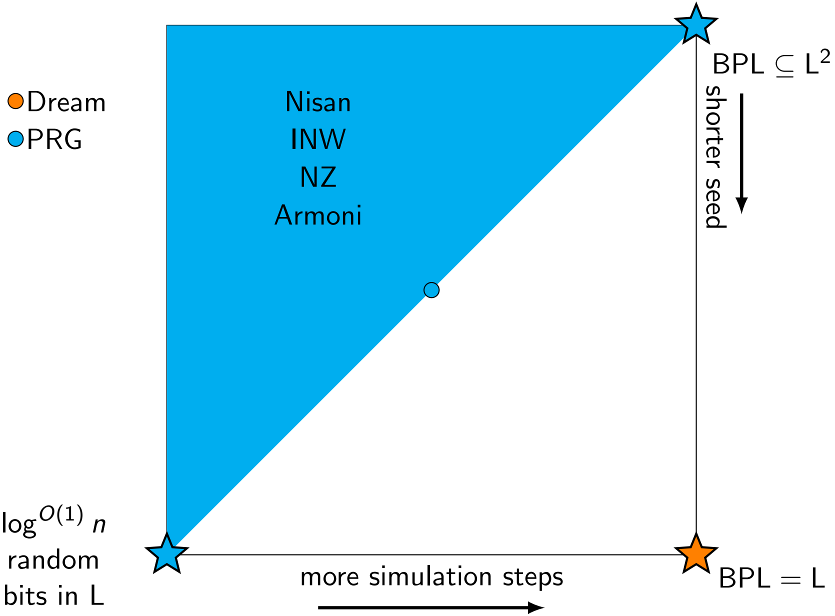

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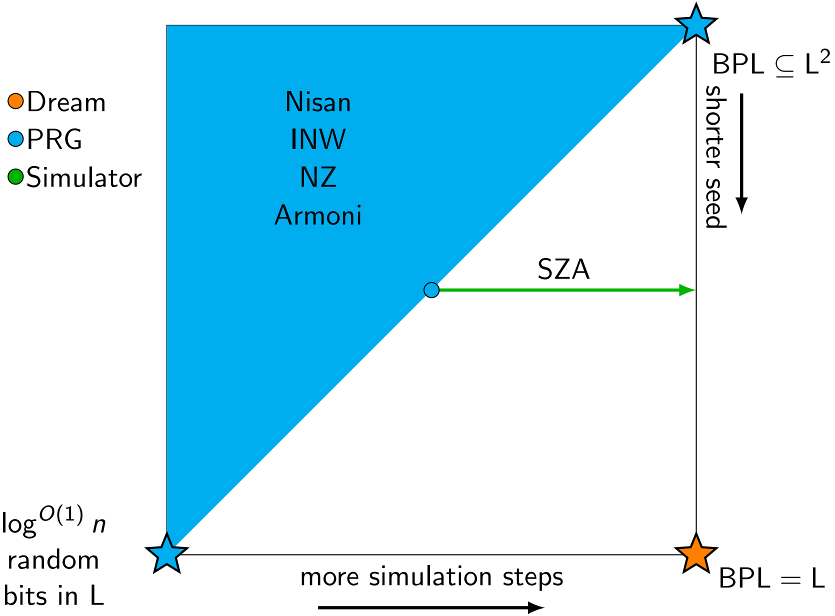

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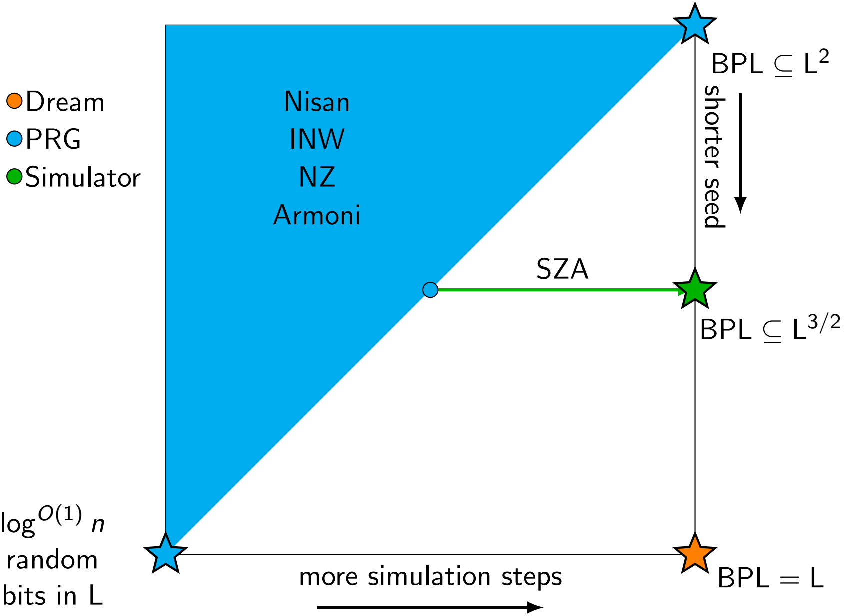

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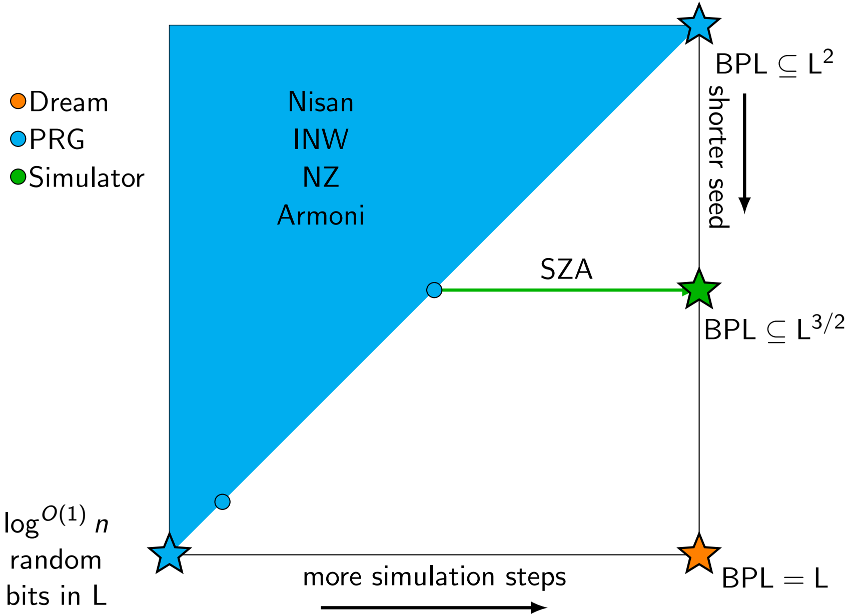

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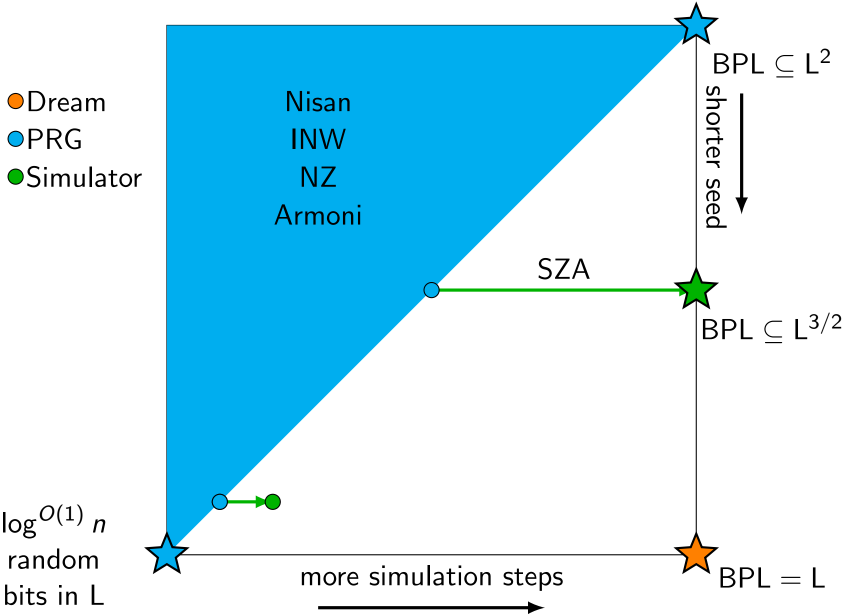

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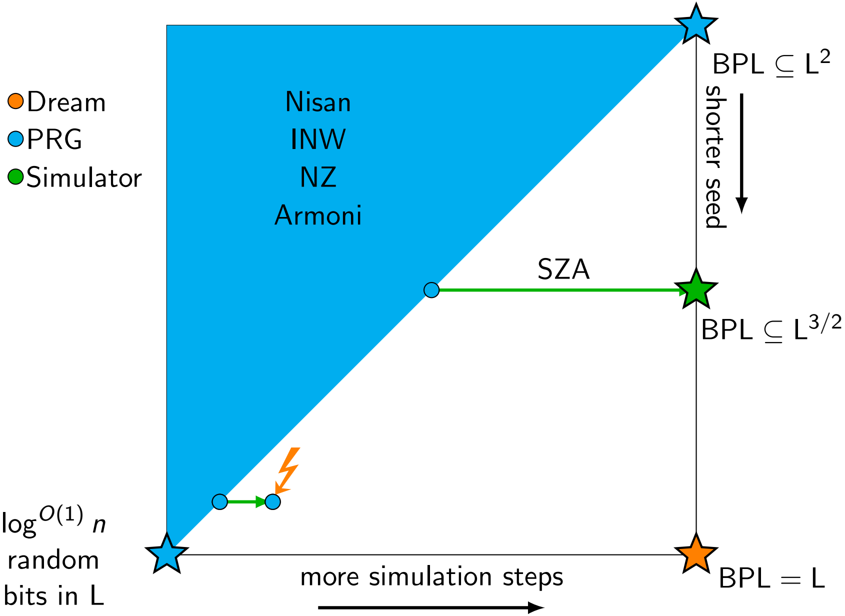

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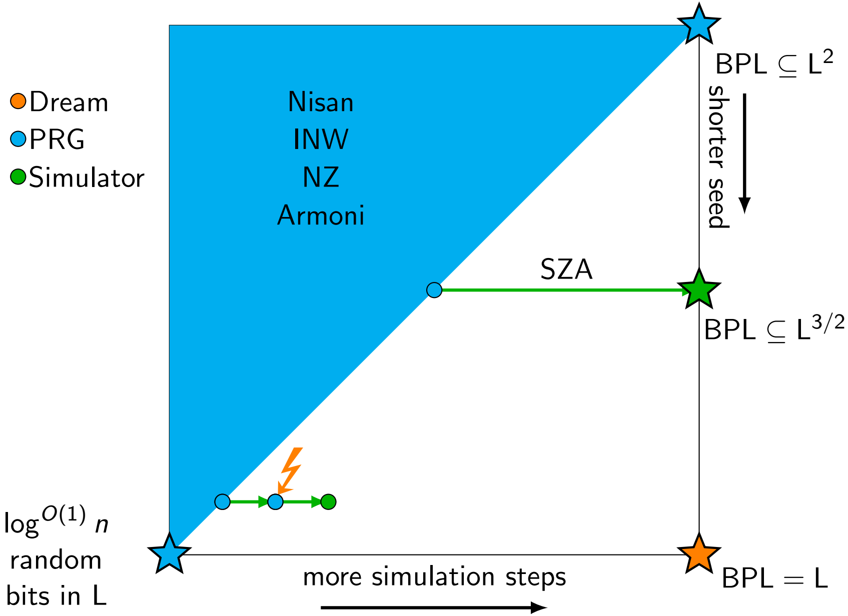

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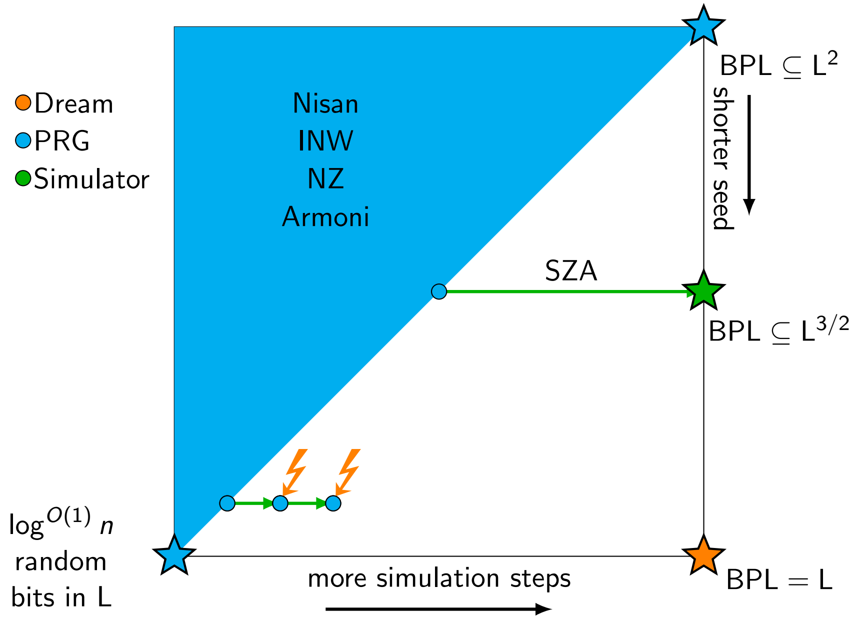

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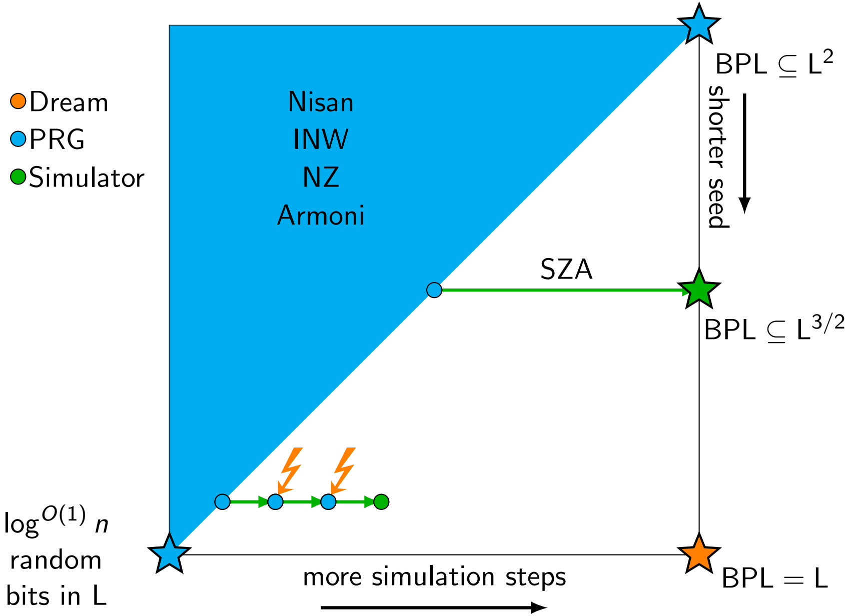

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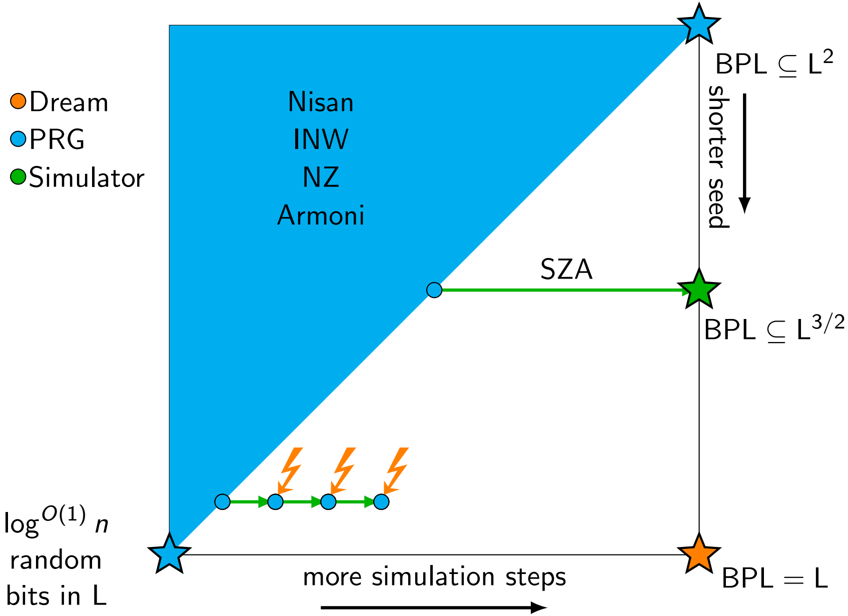

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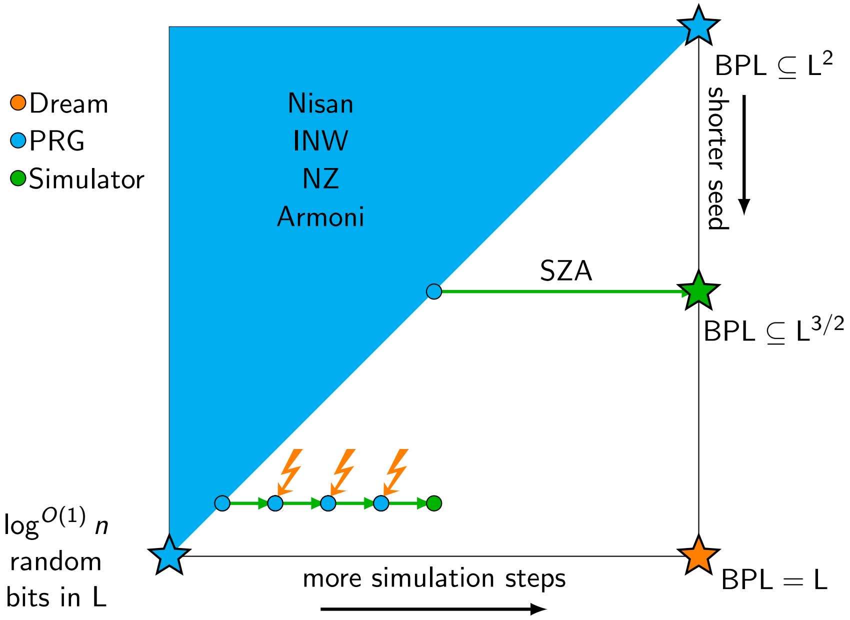

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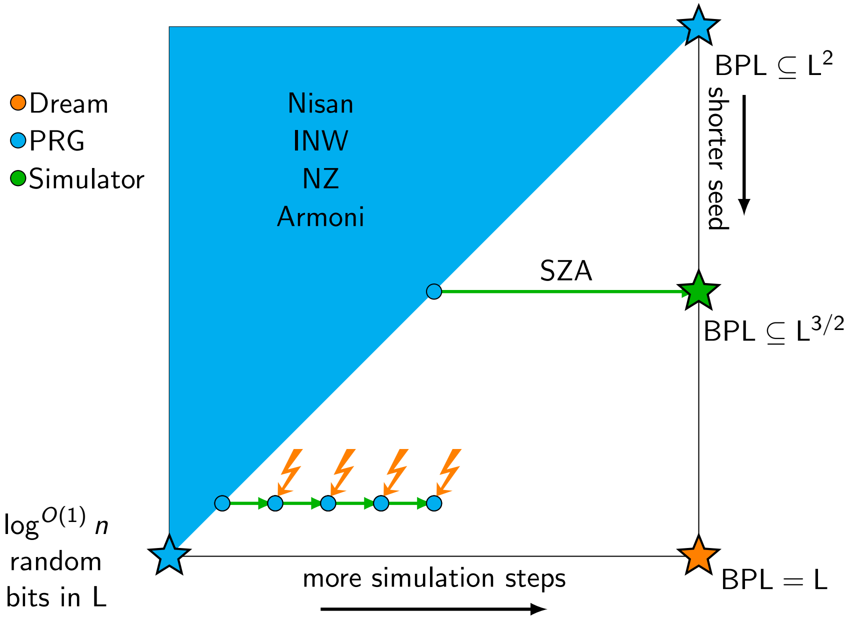

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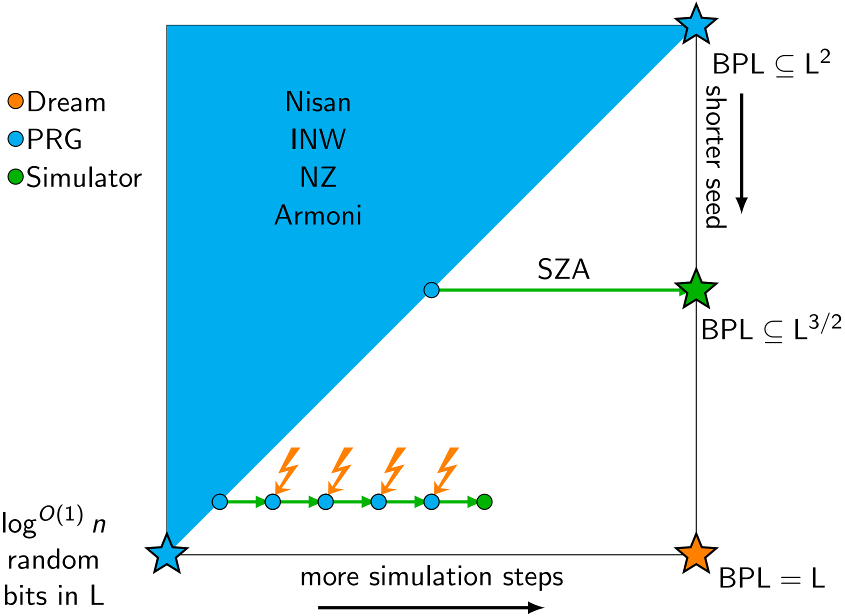

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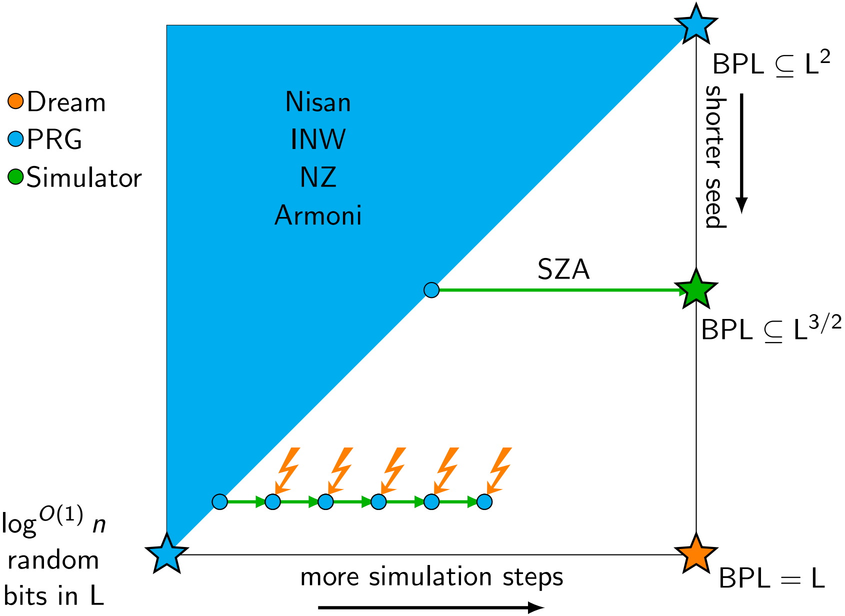

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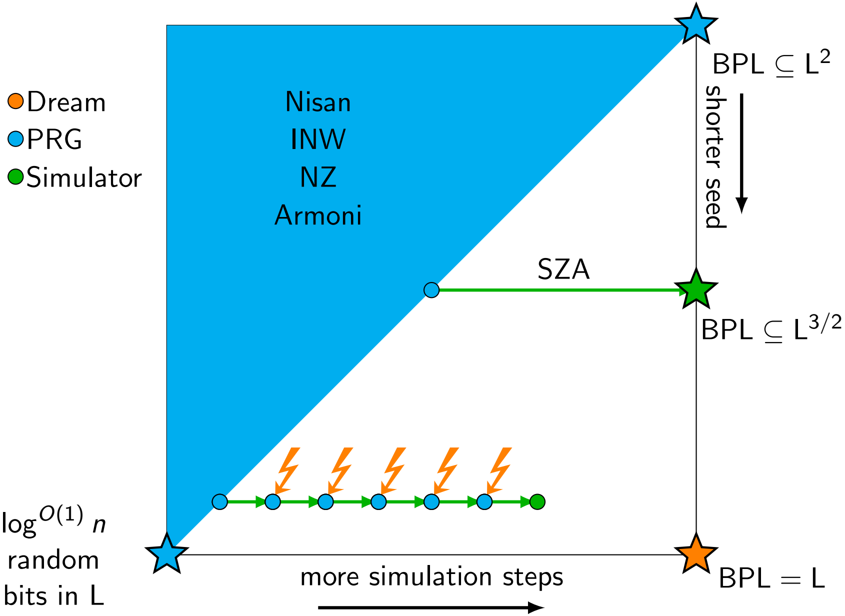

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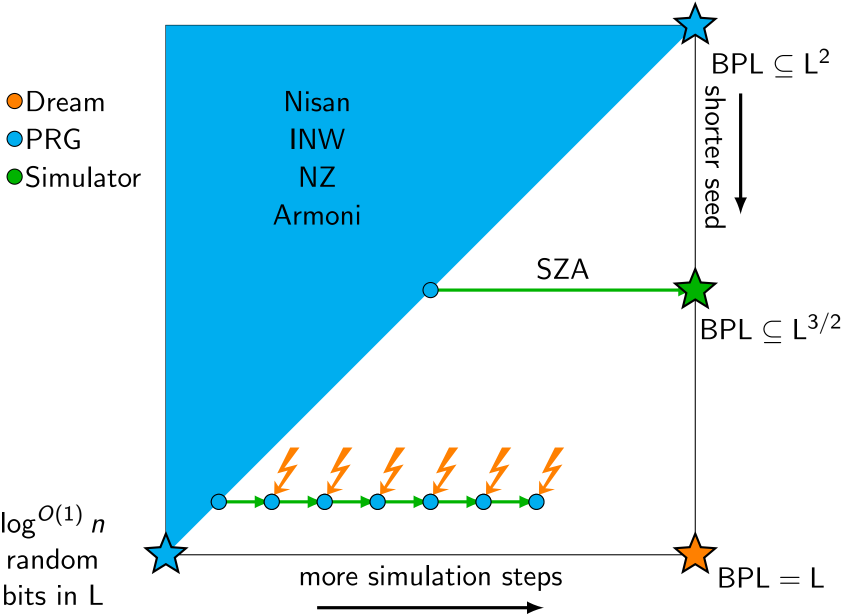

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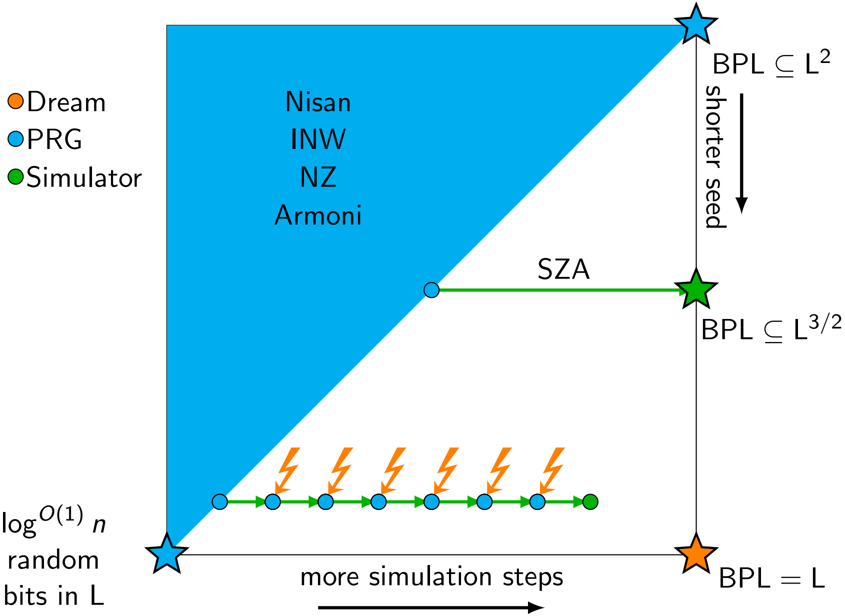

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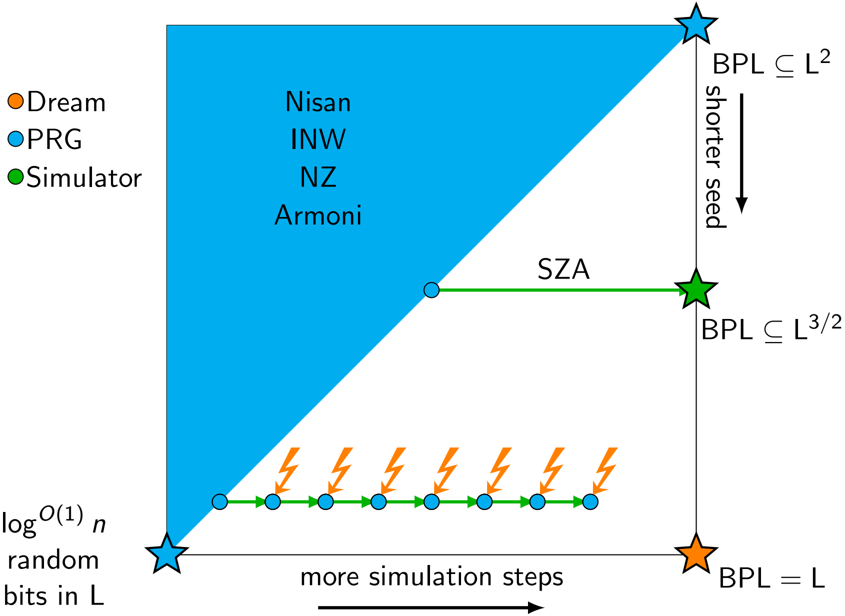

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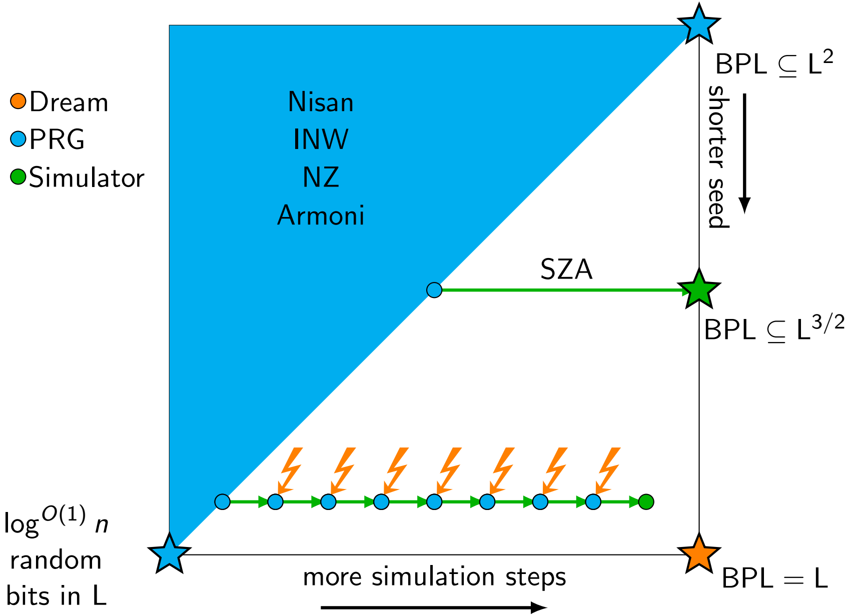

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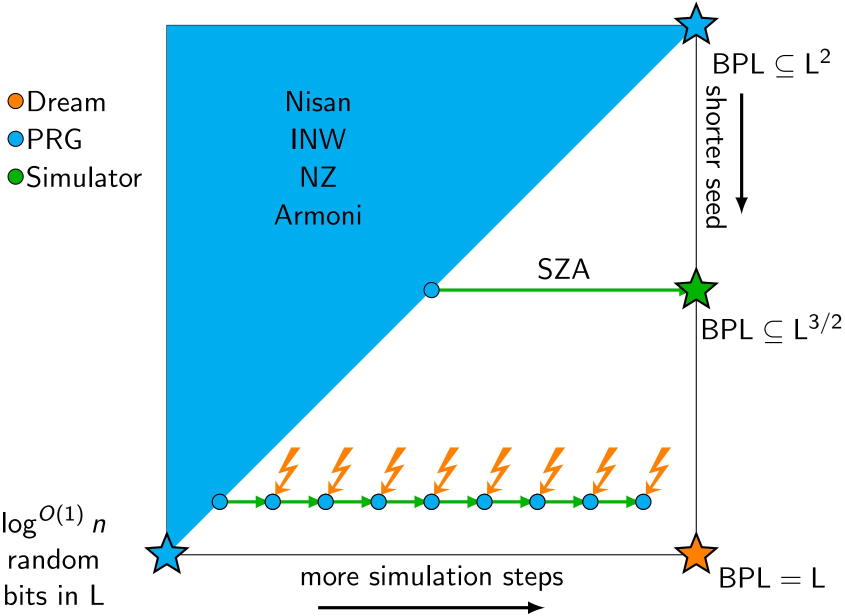

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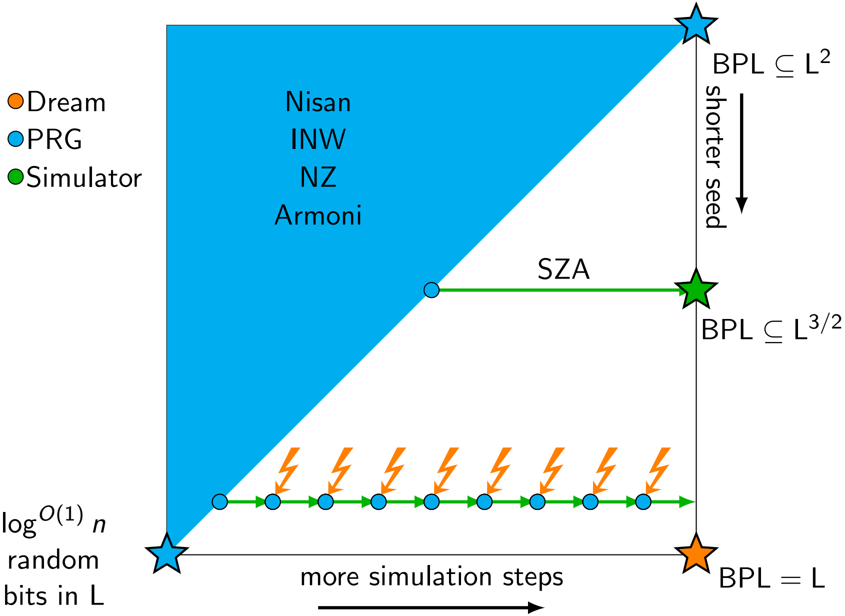

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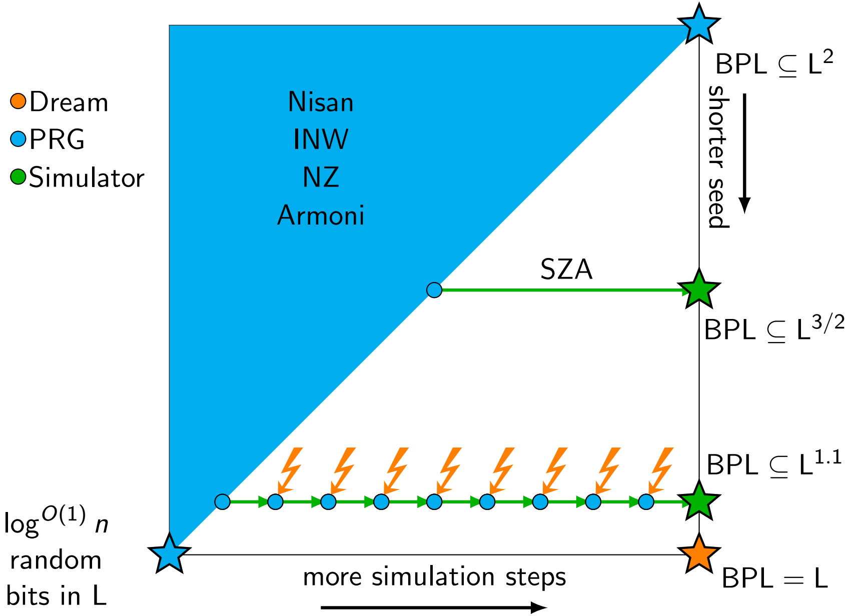

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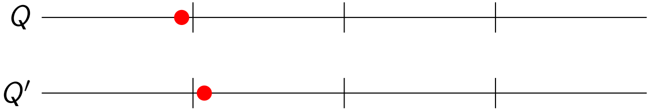

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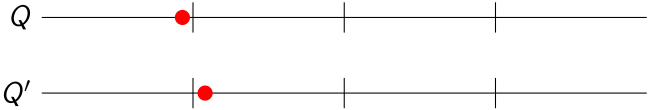

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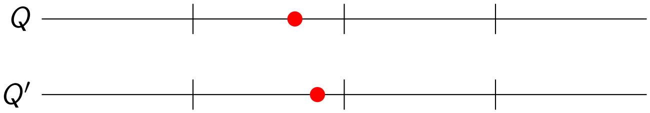

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Outline

- ✓ Simplified statement of main result
- ✓ Proof sketch of main result
  - ✓ Saks-Zhou theorem, revisited
- ✓ Proof sketch of Saks-Zhou-Armoni theorem
- Stronger version of main result
  - Targeted PRGs
  - Simulation advice generators

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Four kinds of derandomization

$$
\begin{tikzcd}
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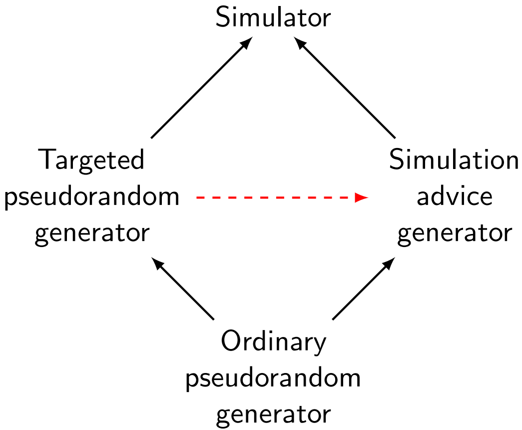
Main result (informally)



Theorem: Dashed arrow transformation exists if and only if

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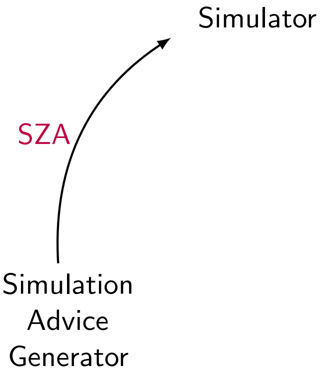

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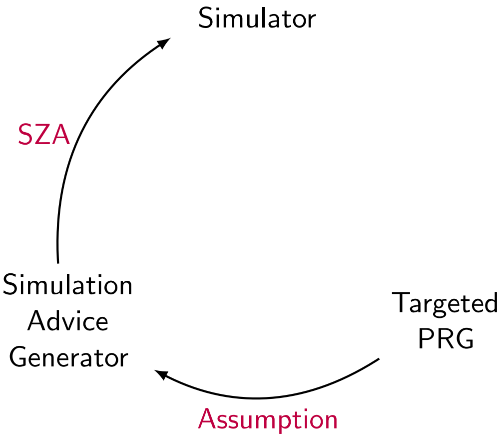

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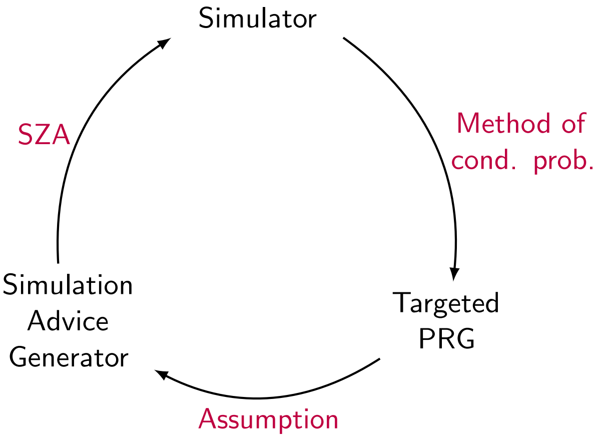

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- "Efficient": Space complexity $\leq O$(seed length)

Conclusion

- This material is based upon work supported by
  - NSF GRFP Grant No. DGE-1610403
  - NSF Grant No. NSF CCF-1423544
- Thanks for your attention!
- Any questions?