Preserving Randomness for Adaptive Algorithms

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Caltech Theory of Computing Seminar

Randomized estimation algorithms

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- Canonical example:
  - C is a Boolean circuit
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  - Est(C) evaluates C at several randomly chosen points

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- **Theorem**: Can use just $n + O(k \log(d+1))$ random bits!
  - Slight increases in error, failure probability

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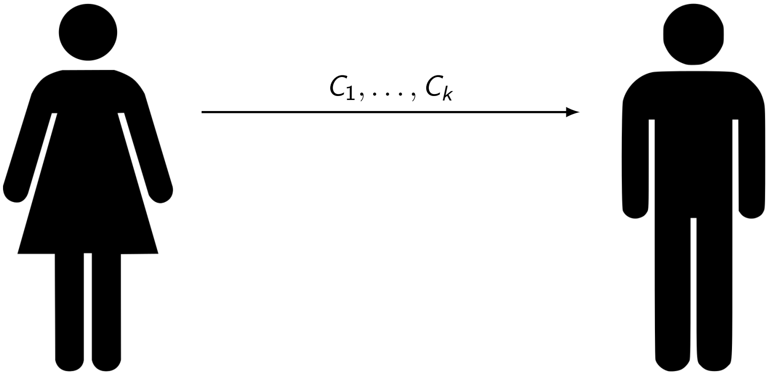

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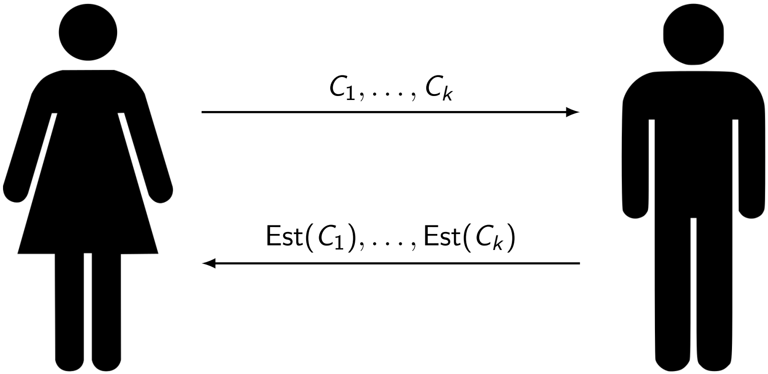

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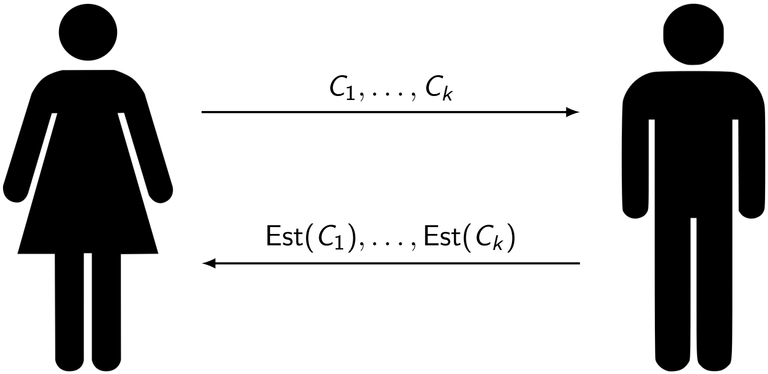

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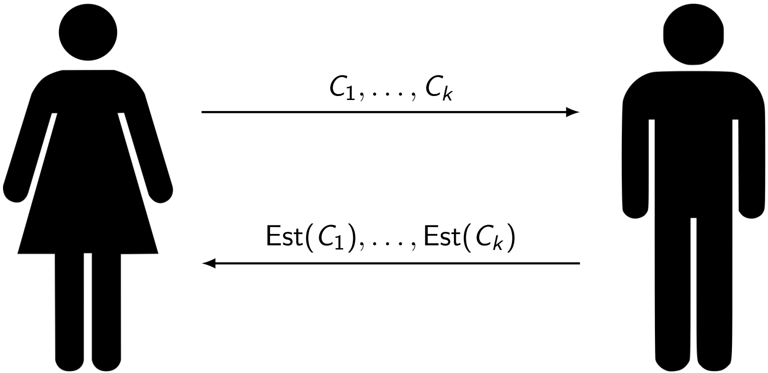

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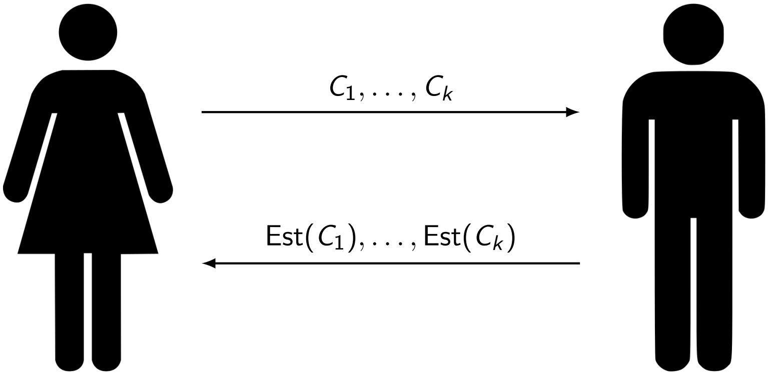

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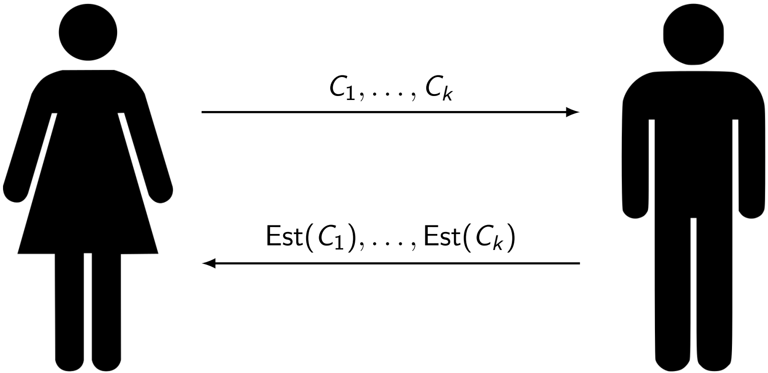

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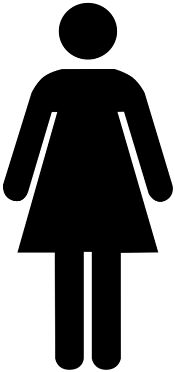
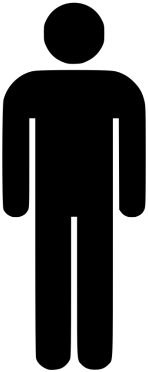
- Overall failure probability is still $k\delta$ (union bound)

Adaptive setting





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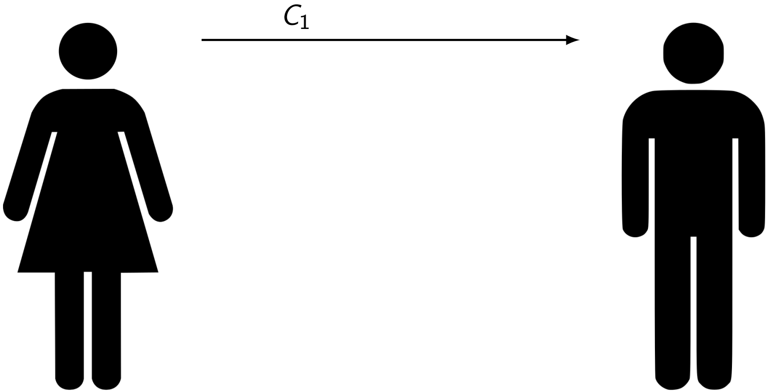

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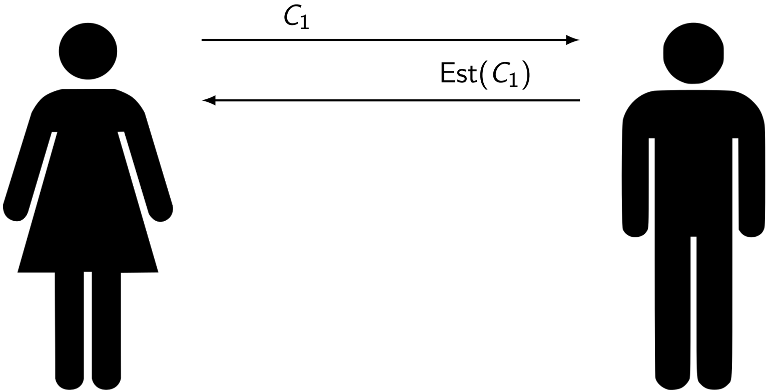

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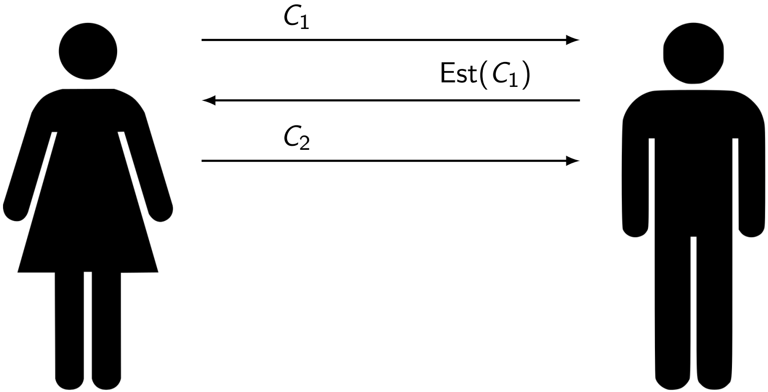

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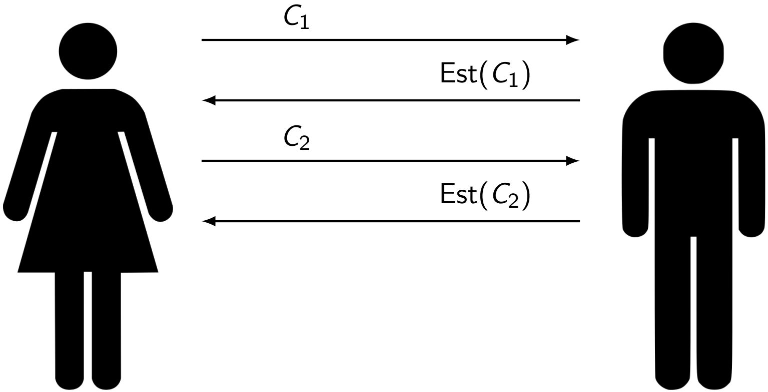

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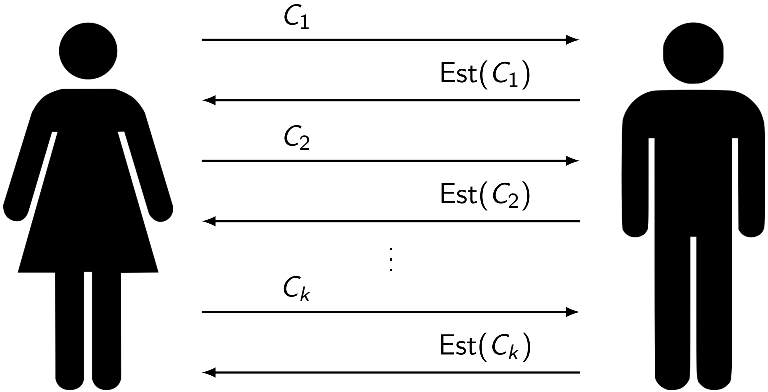

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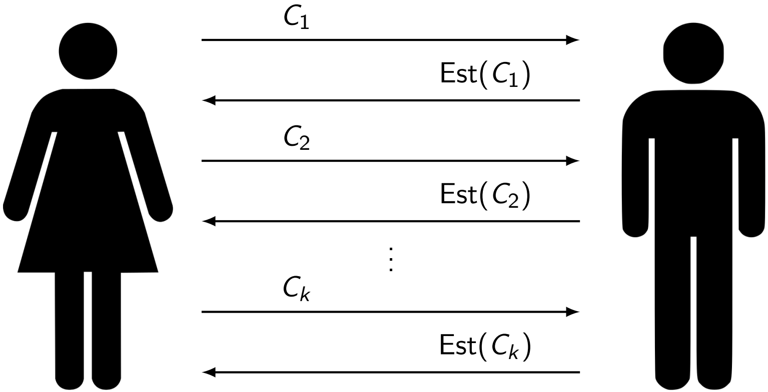

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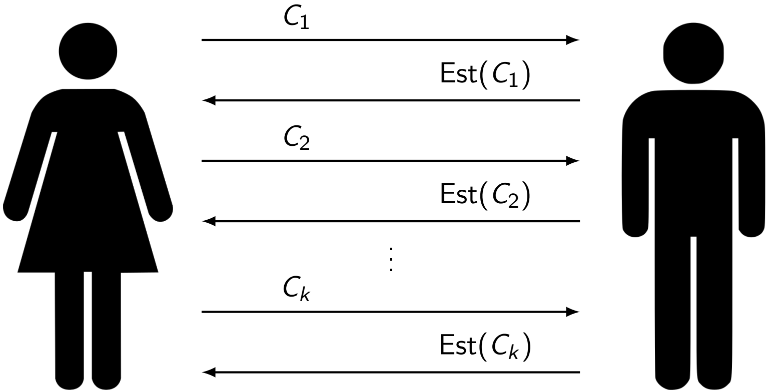

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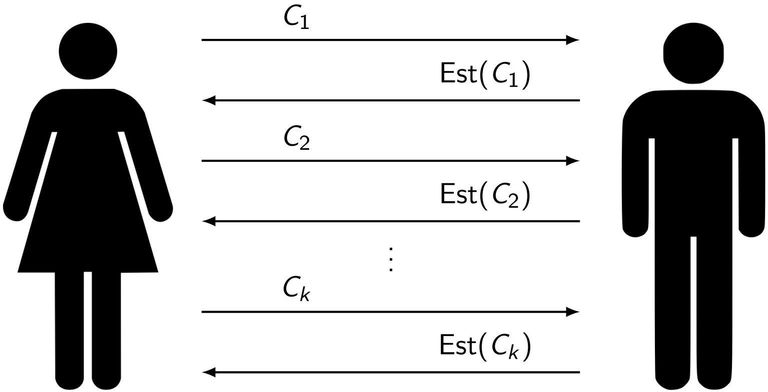

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- C_2 is stochastically dependent on X
- Failure probability of $\mathrm{Est}(C_2, X)$ is ???

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- Example: $f(X) \stackrel{\text{def}}{=} \text{Est}(C, X)$

Randomness steward model

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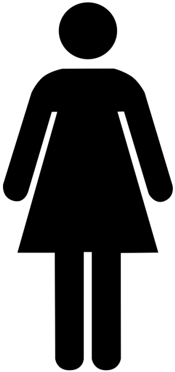
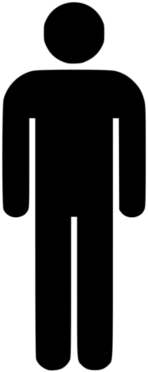


Steward



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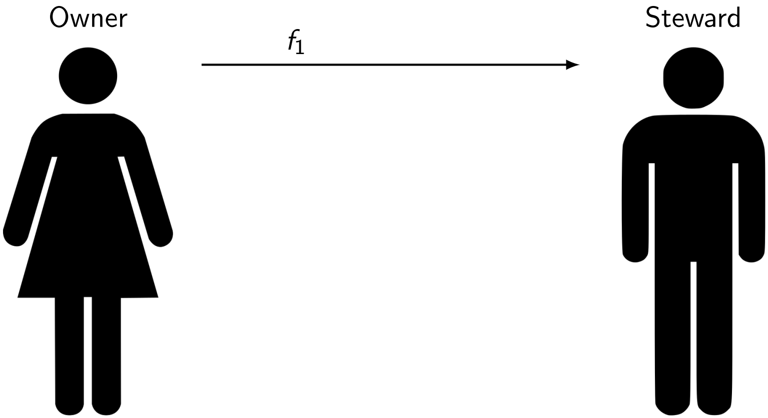

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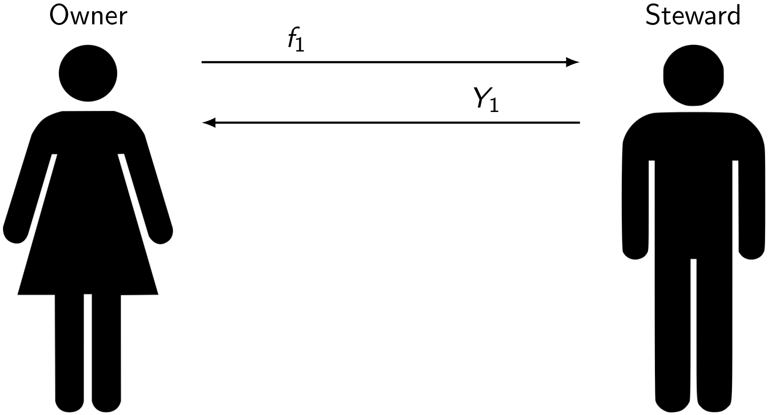

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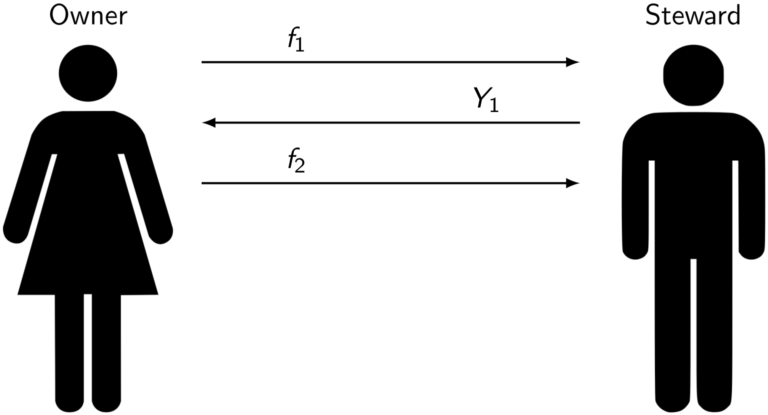

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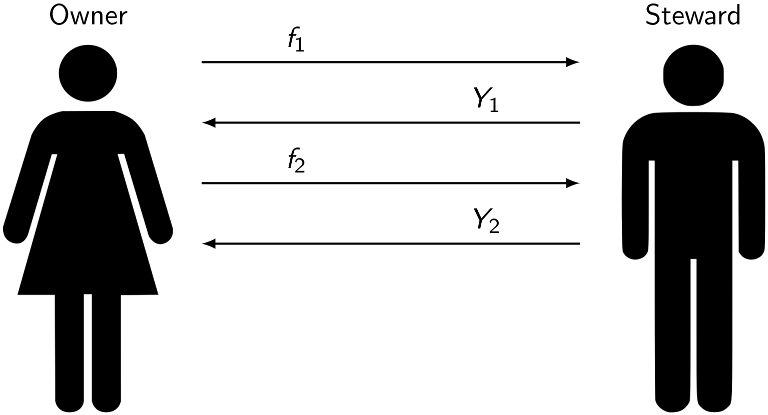

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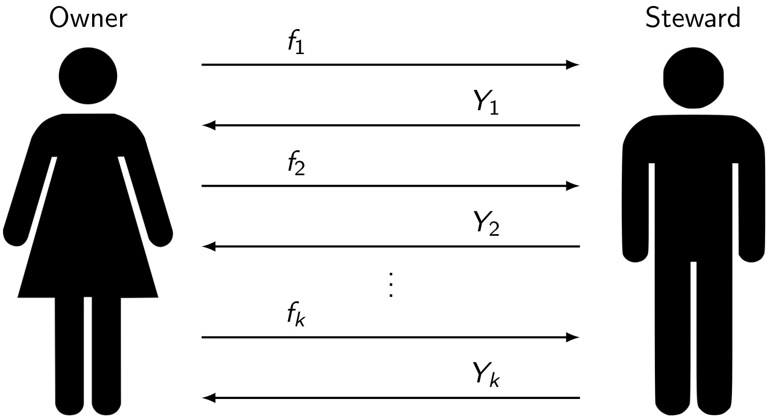

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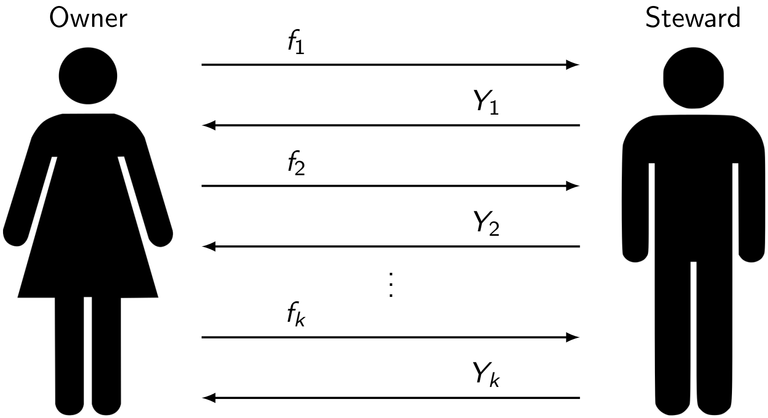

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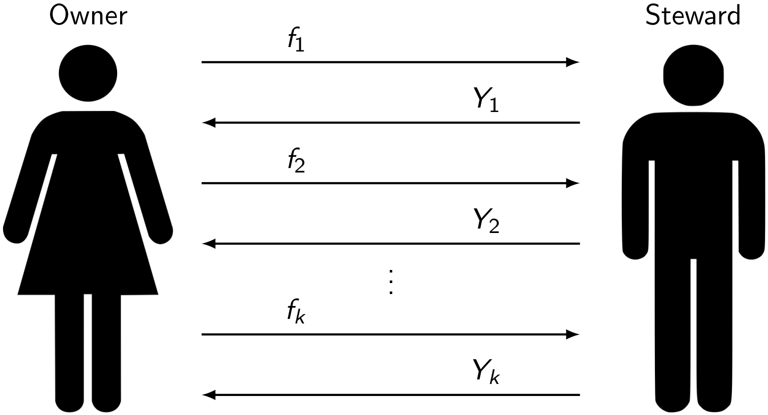

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- Steward requirement: For any owner,

$$\Pr\left[\max_i \| Y_i - \mu_i \|_\infty > \varepsilon'\right] \leq \delta'$$

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  - For $i = 1$ to k: Return $f_i(X)$, rounded to multiple of 2ε

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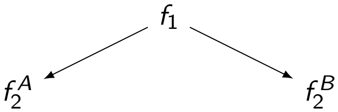

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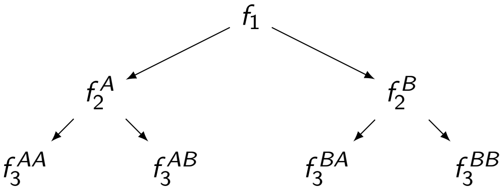

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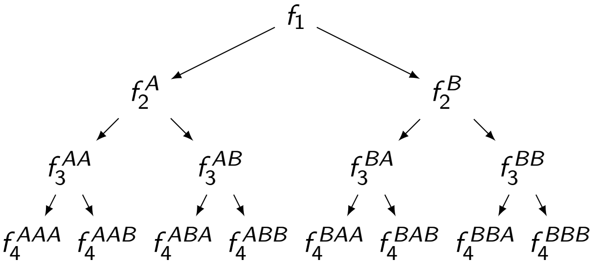

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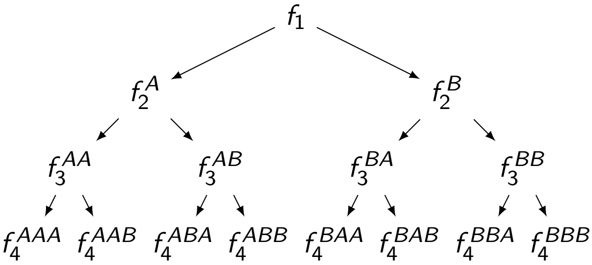

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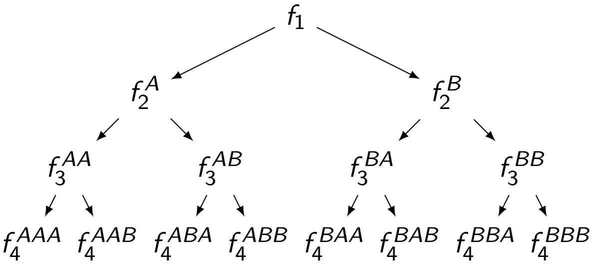

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- If so, inductively, every f_i is in the tree!

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  - # random bits $n + O(k \log(d+1) + \log k \log(1/\gamma))$

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- Ingredient 2: Deterministic shifting and rounding algorithm

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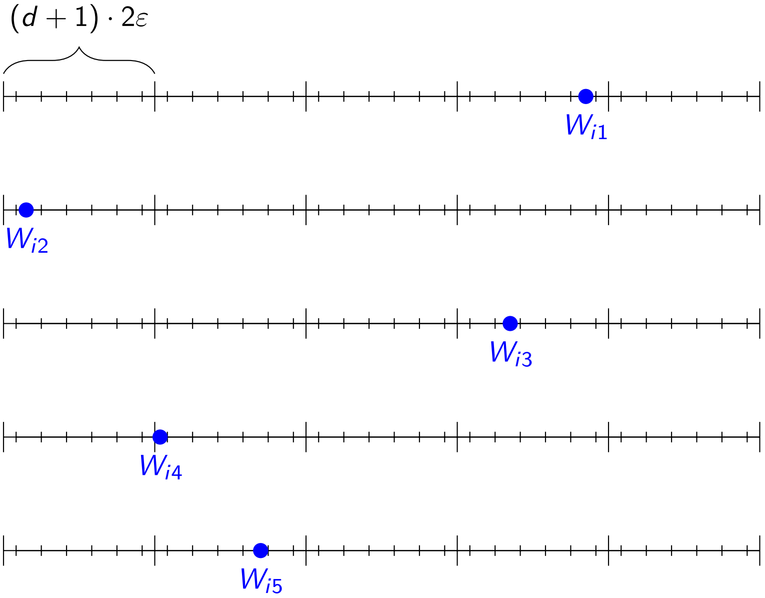

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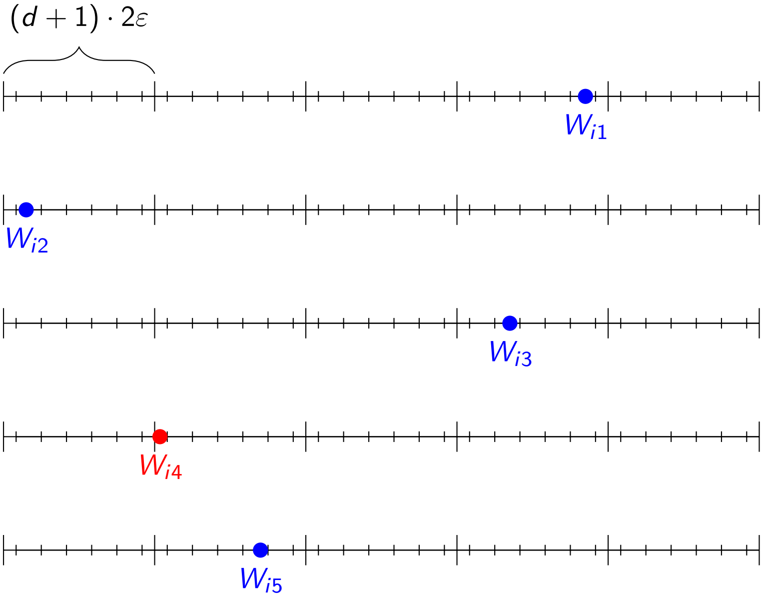

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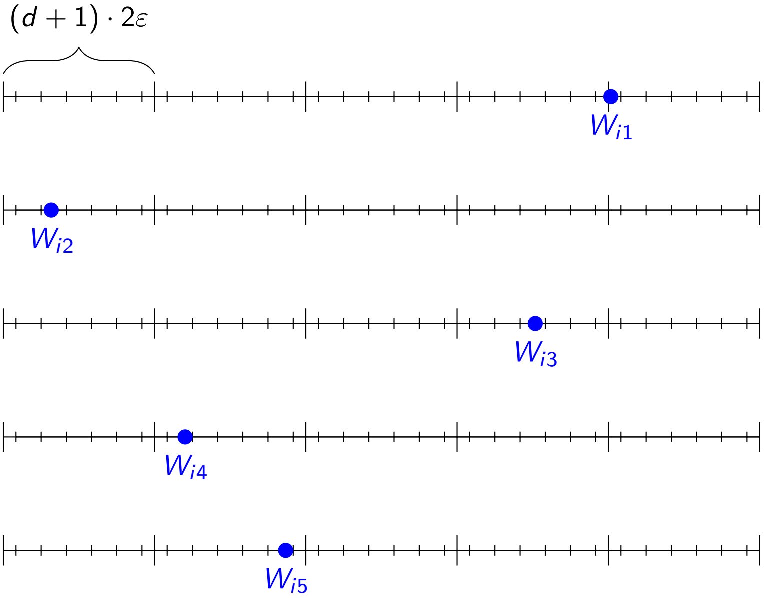

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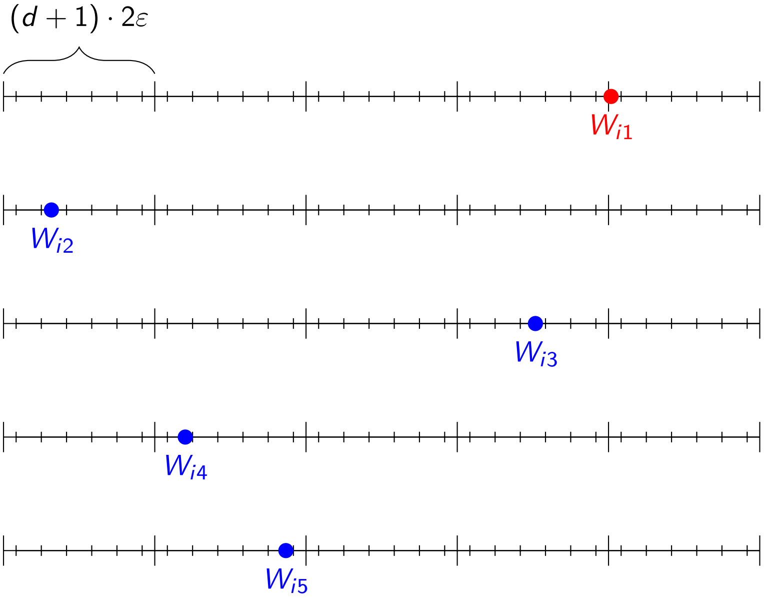

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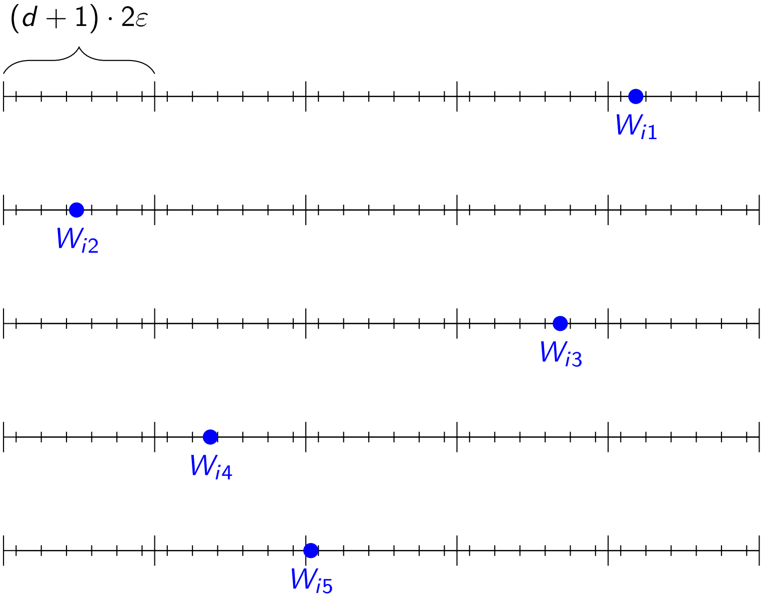

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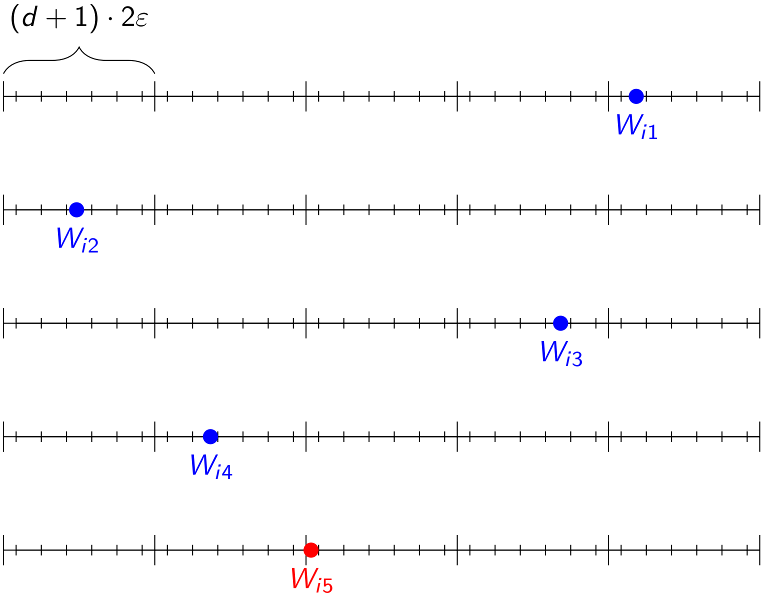

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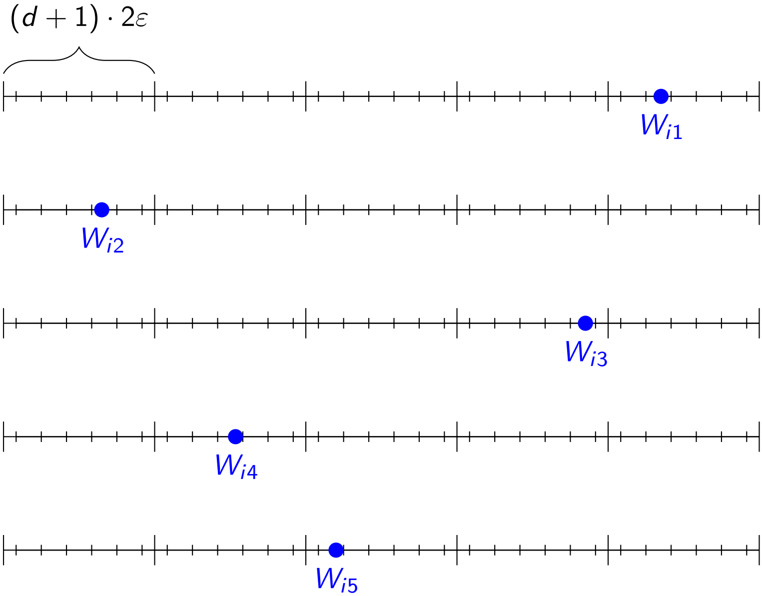

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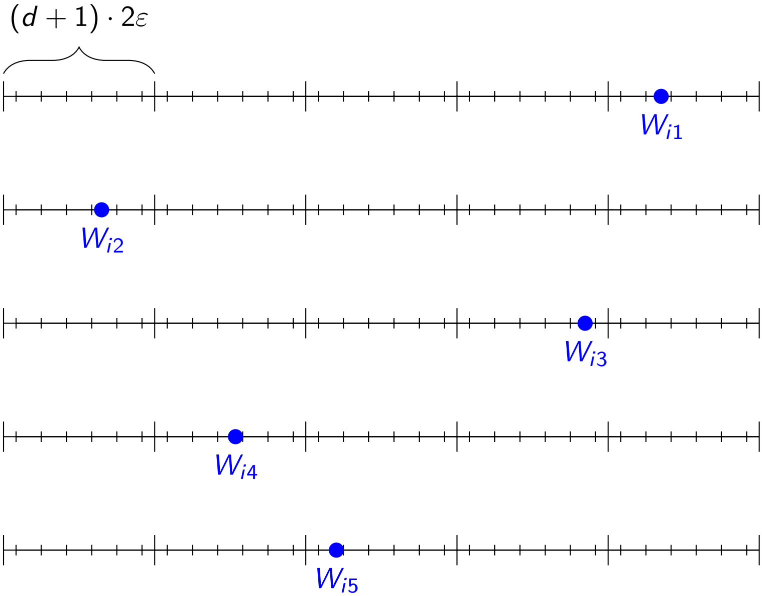

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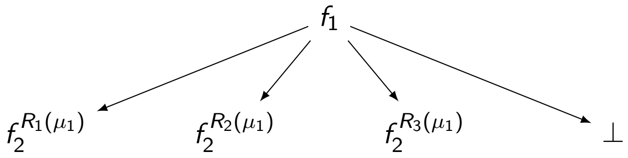

Analysis of shifting and rounding algorithm

- For $W \in \mathbb{R}^d$ and $\Delta \in [d+1]$, define $R_\Delta(W) \in \mathbb{R}^d$ by shifting W according to Δ, then rounding
- By construction, $Y_i = R_\Delta(W_i)$ for some Δ
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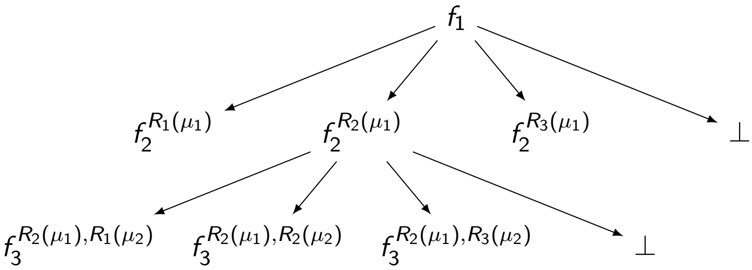
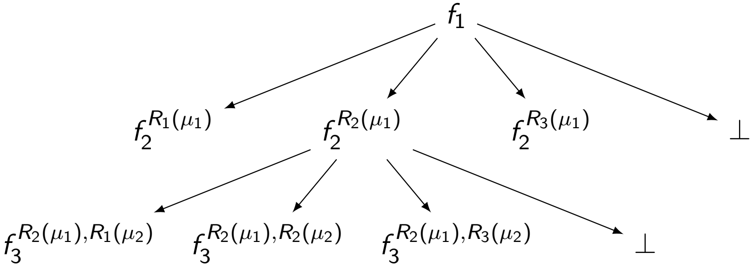


Certification tree



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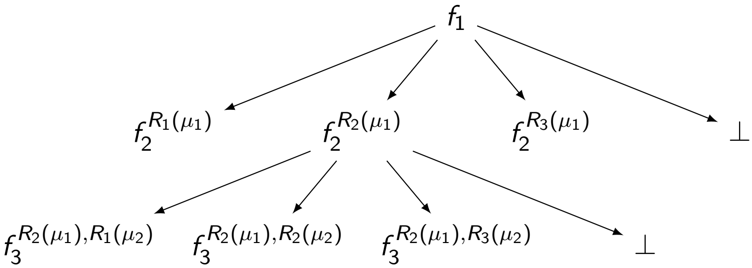

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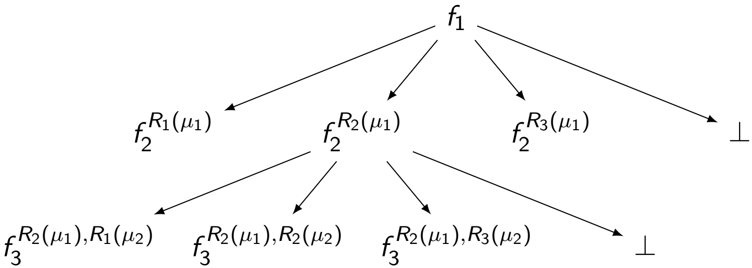

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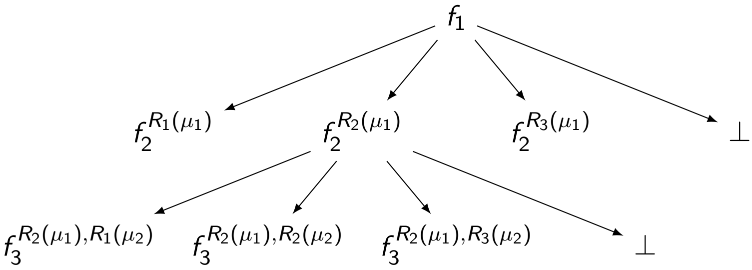

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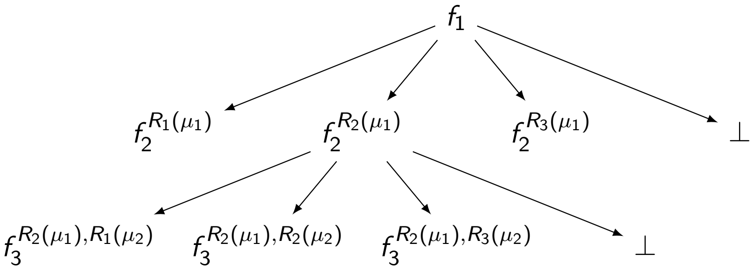

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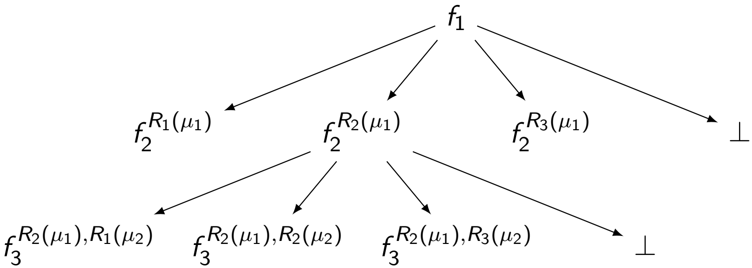

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- (Certification) No $\perp$ nodes in $P \implies$ every Y_i has error $O(\varepsilon d)$

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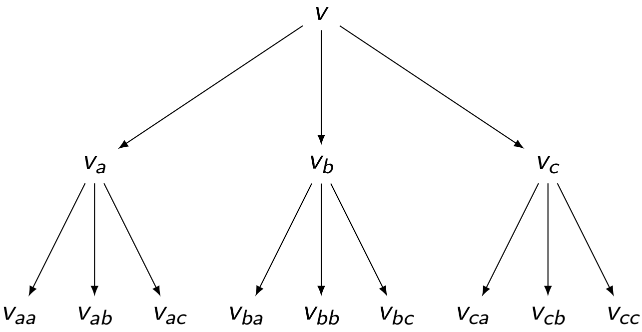

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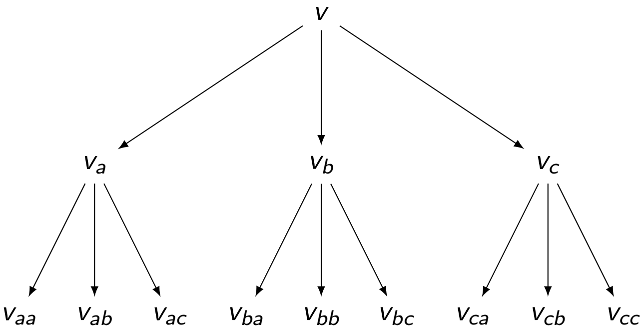

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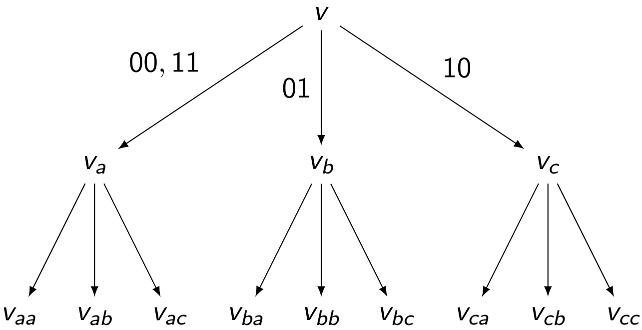

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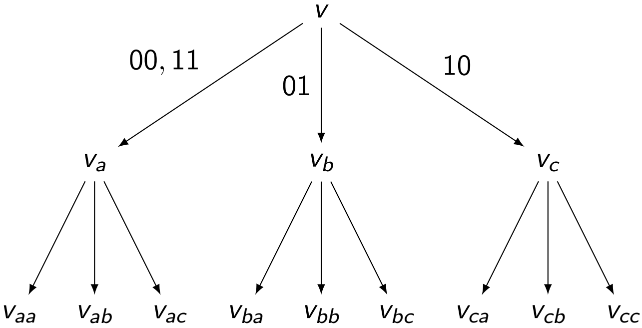

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- Thanks! Questions?
- This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1610403.