Preserving Randomness for Adaptive Algorithms

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RANDOM

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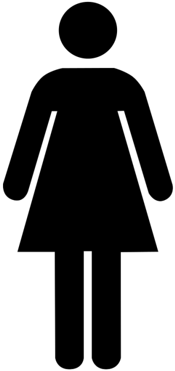
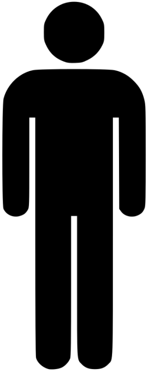
- Canonical example:
  - C is a Boolean circuit
  - $\mu(C) \stackrel{\text{def}}{=} \Pr_x[C(x) = 1] \quad (d = 1)$
  - $\mathrm{Est}(C)$ evaluates C at several randomly chosen points

Using Est as a subroutine

Owner



Steward



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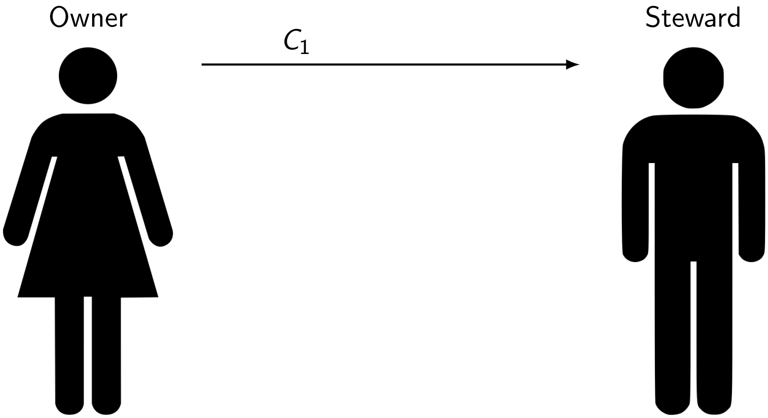

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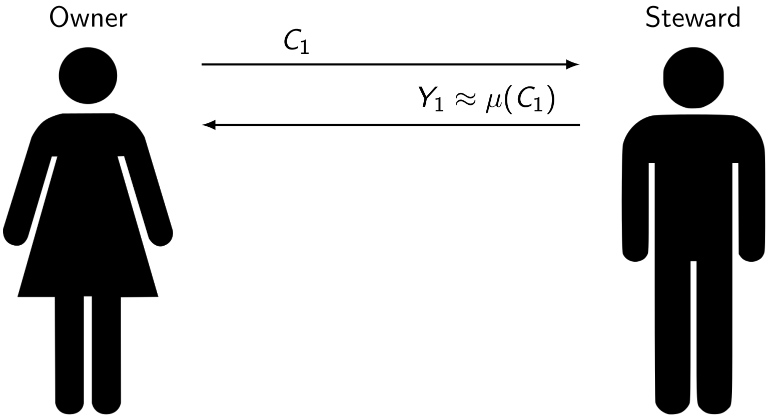

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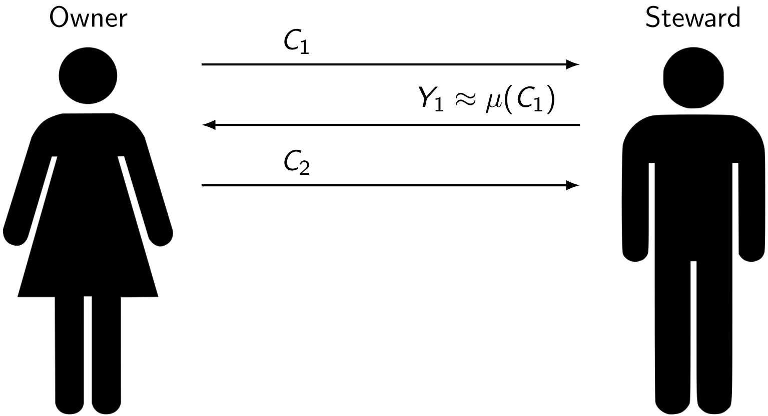

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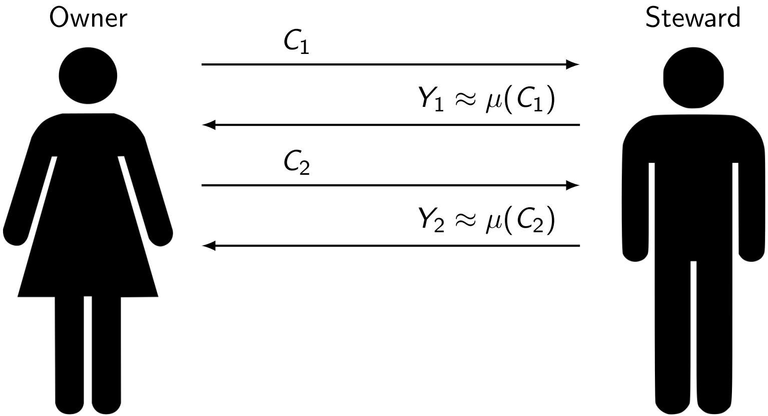

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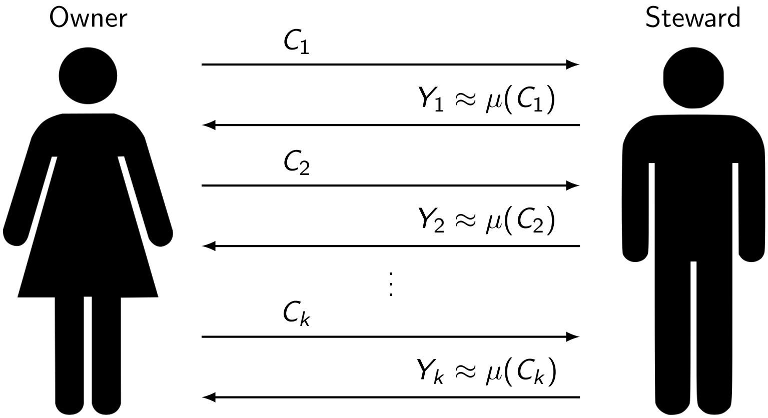

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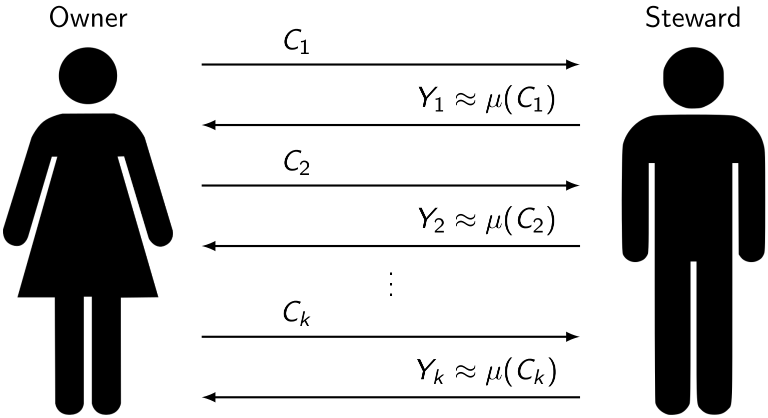

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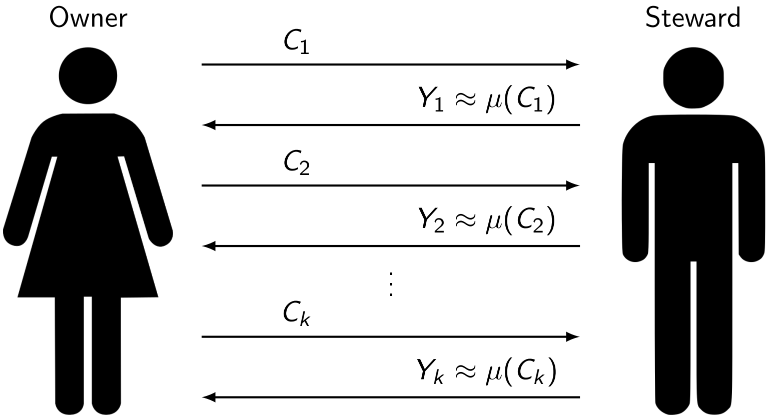

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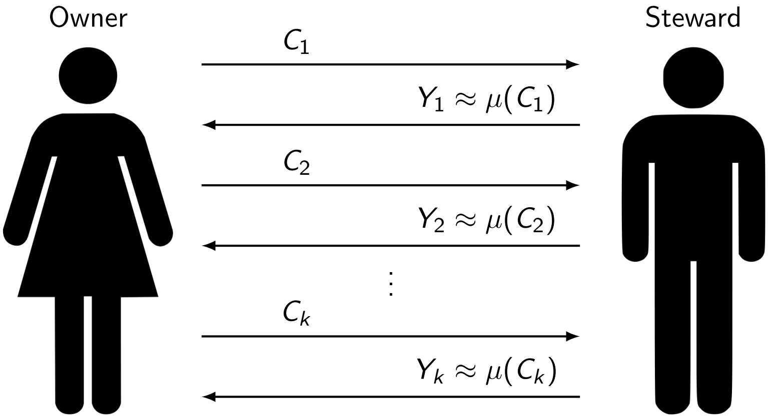

- Suppose Est uses n random bits
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- **Can we do better?**

Main result

- **Theorem** (informal): There is a steward that uses just

$$n + O(k \log(d+1))$$

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  - Our steward has better parameters

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3. Compute Y_i by carefully modifying W_i

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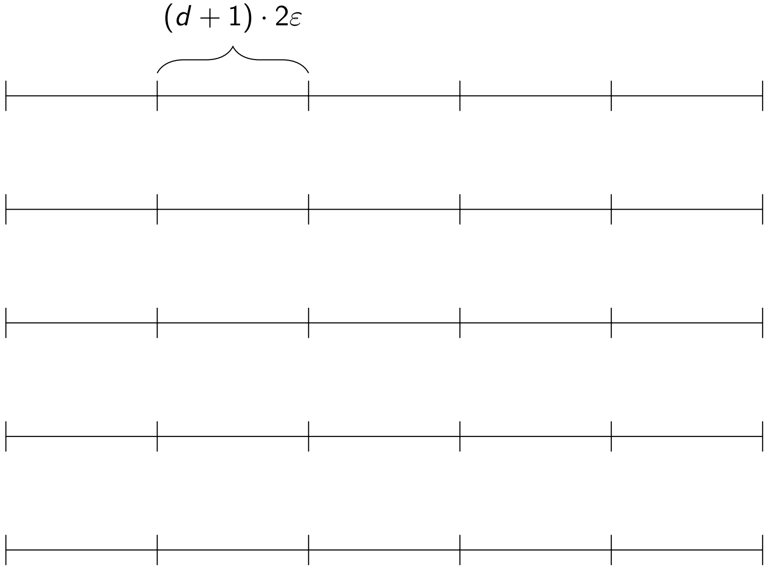
- In round i, steward runs $\mathsf{Est}(C_i, X_i)$

Shifting and rounding



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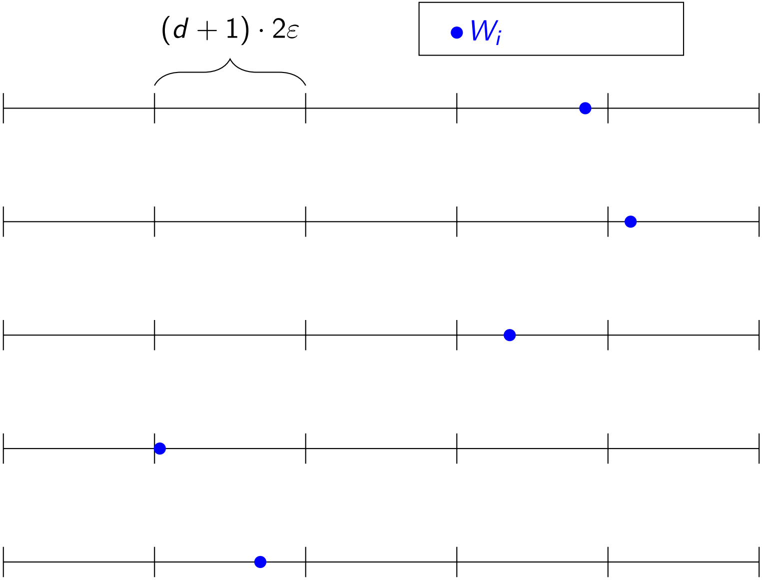

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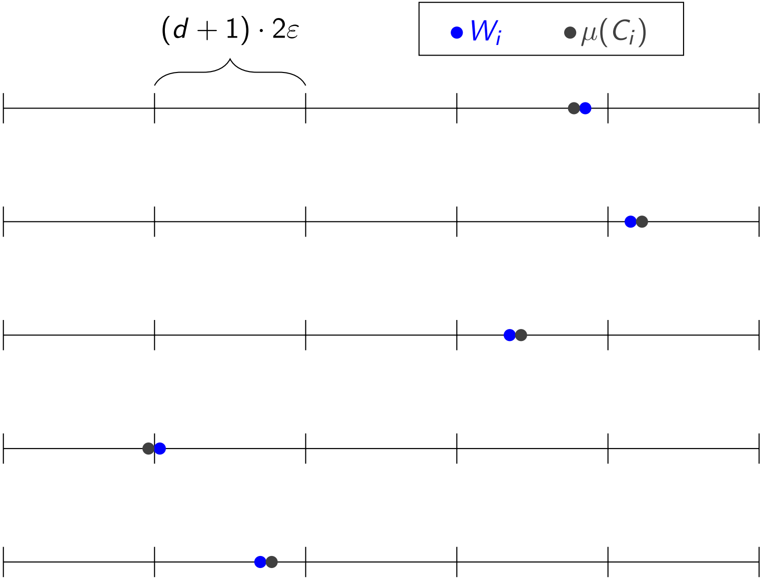

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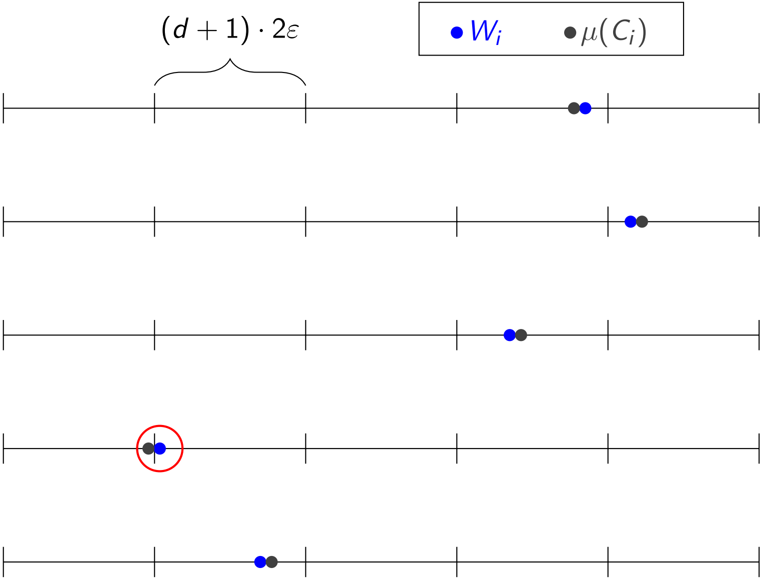

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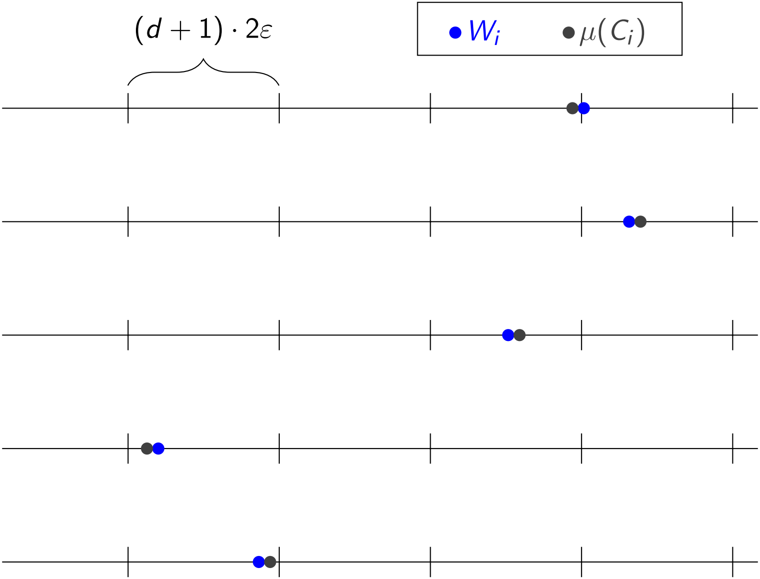

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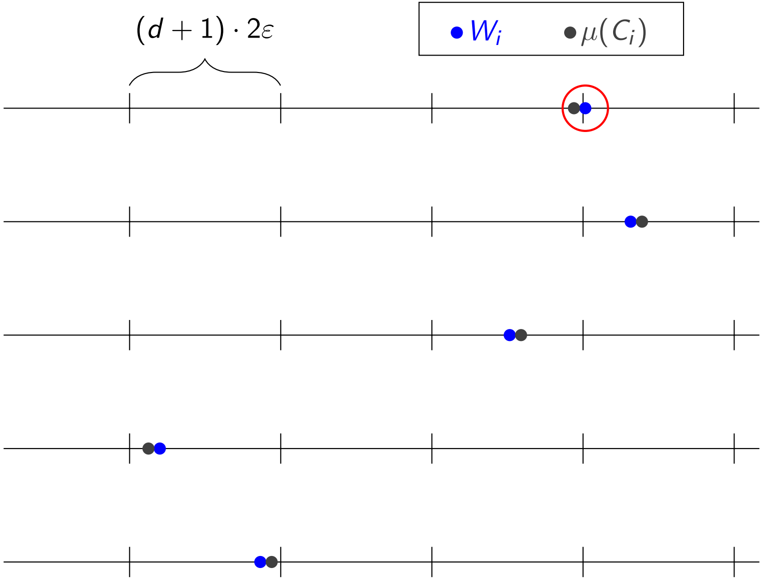

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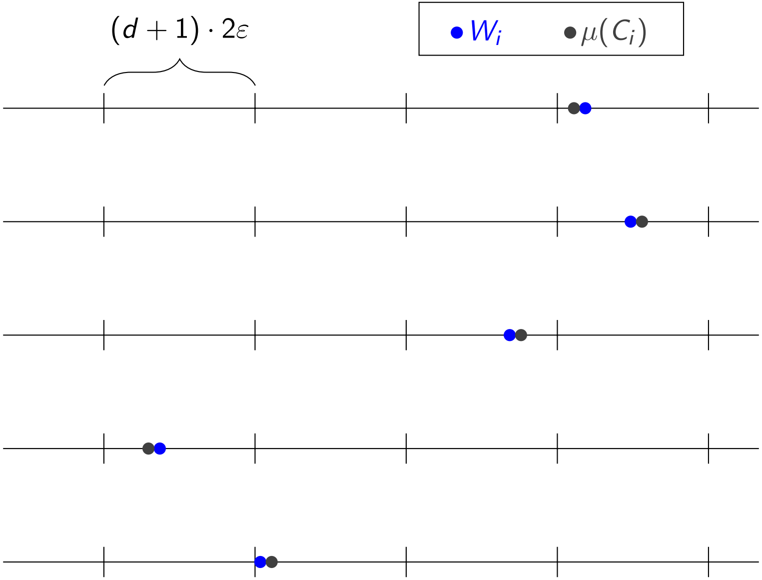

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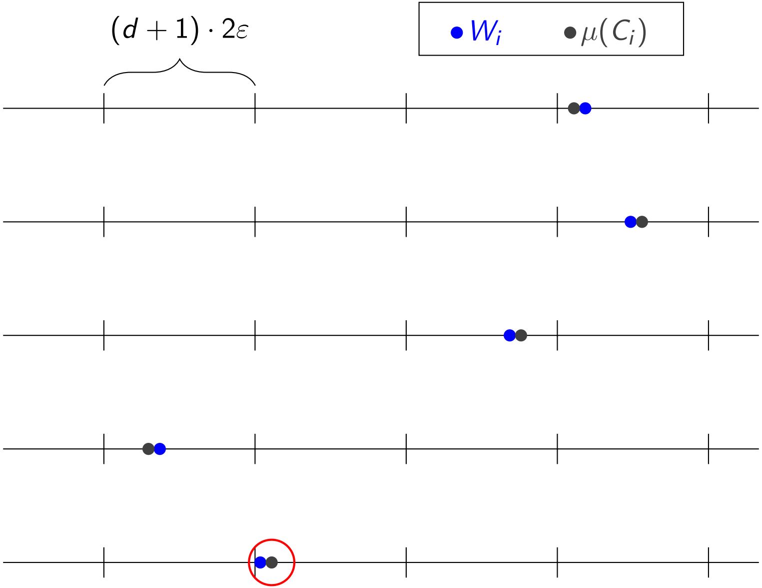

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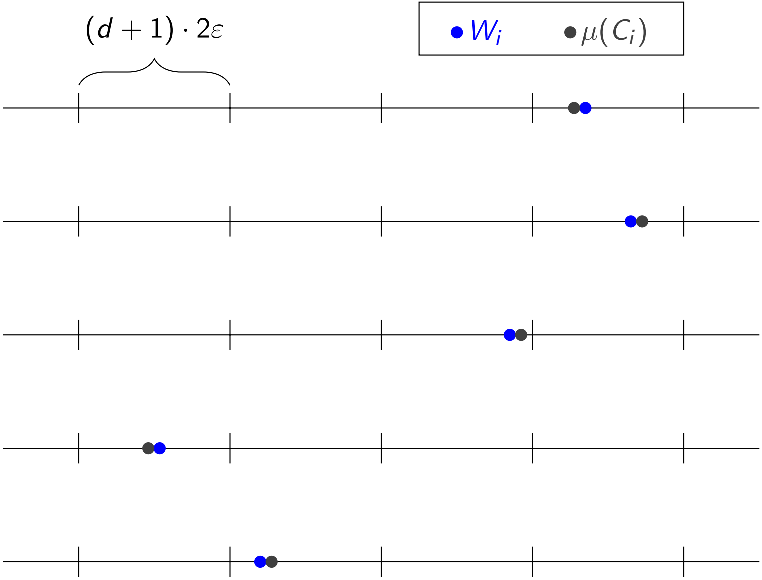

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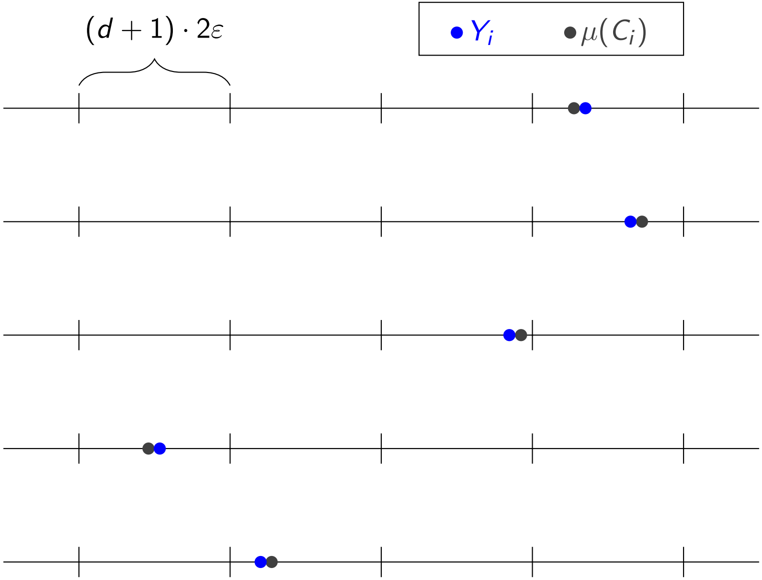

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- In this notation,

$$Y_i = \lfloor W_i \rceil_\Delta$$

  for a suitable $\Delta \in [d+1]$

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  - $Y_i = \lfloor \mu(C_i) \rceil_{d+1}$, or
  - Y_i = something else.

Block decision tree

C_1

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$$\begin{tikzcd}
 & & & C_1 \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
C_2^{(1)} & & C_2^{(2)} & & C_2^{(3)} & & \perp
\end{tikzcd}$$

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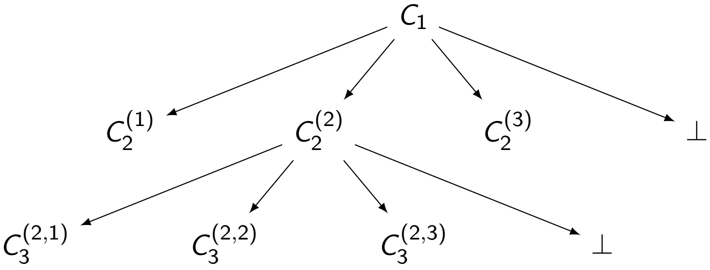

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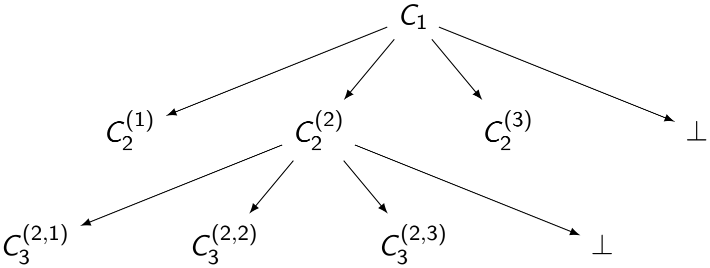

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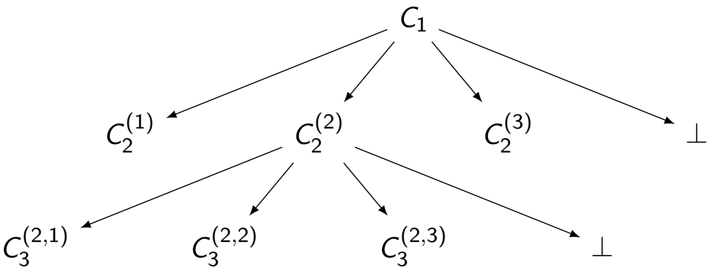
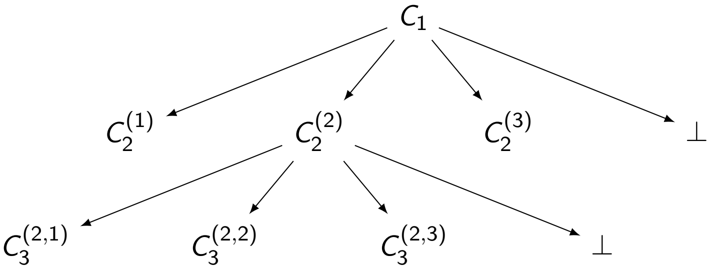


- A sequence $(X_1, \ldots, X_k)$ determines:
  - A transcript $(C_1, Y_1, C_2, Y_2, \ldots, C_k, Y_k)$
  - A path P through tree
- If we pick $X_1, \ldots, X_k$ independently and u.a.r.,

$$\Pr_{(X_1, \ldots, X_k)}[P \text{ has a } \perp \text{ node}] \leq k\delta$$

Fooling the tree



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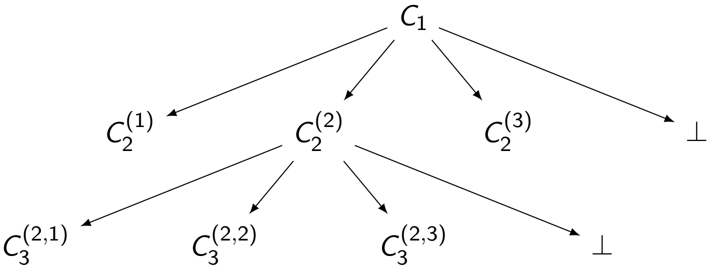

- Tree has low memory
- So when $X_1, \ldots, X_k$ are pseudorandom,

$$\Pr_{(X_1,\ldots,X_k)}[P \text{ has a } \perp \text{ node}] \leq k\delta + \gamma$$

The tree certifies correctness

$$
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- (Certification) No $\perp$ nodes in $P \implies$ every Y_i has error $O(\varepsilon d)$

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  - Goldreich-Wigderson sampler

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