Depth-d Threshold Circuits vs. Depth-$(d+1)$ AND-OR Trees

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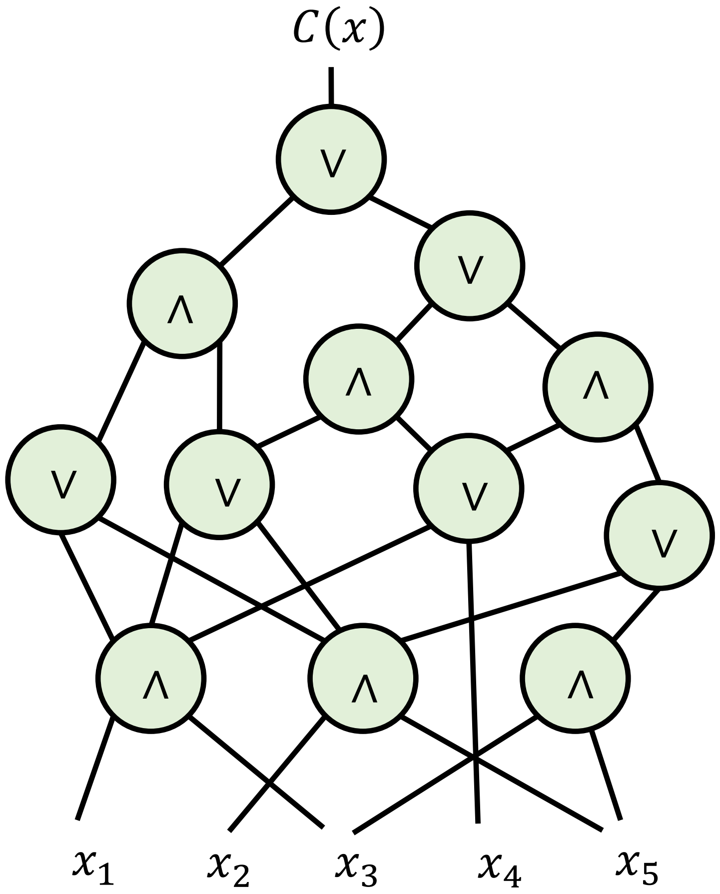
Circuit lower bounds

- $\exists F: \{0,1\}^n \to \{0,1\}$ with circuit complexity $2^{\Omega(n)}$
- Challenge: Find explicit examples
- TCS model: "Explicit" ≡ "Efficiently computable"
- So, we want to construct F such that
  - The circuit complexity of F is high…
  - …but F can be computed by a uniform algorithm that is as efficient as possible
- Example: If $\exists F \in \mathrm{NP}$ with super-polynomial circuit complexity, then $\mathrm{P} \neq \mathrm{NP}$



State of the art

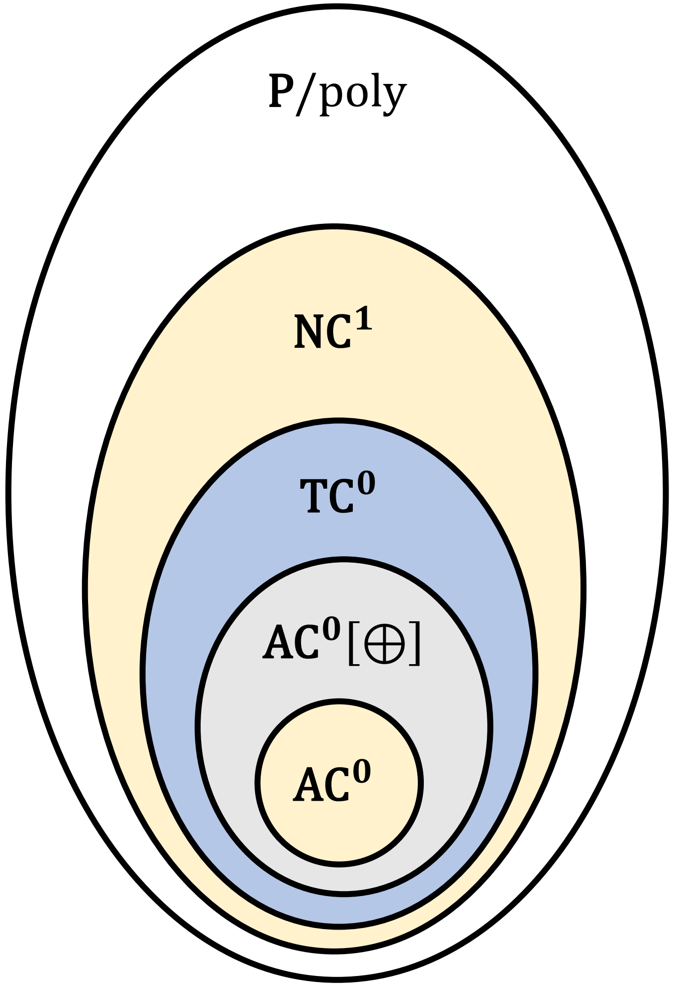
- So far, the best circuit lower bounds for functions in NP are only linear ☹

Basis	Size lower bound	Reference
B_2	$2n - O(1)$	[Kloss, Malyshev 1965]
B_2	$2.5n - O(1)$	[Stockmeyer 1977]
B_2	$3n - o(n)$	[Blum 1984]
B_2	$\left(3 + \frac{1}{86}\right) n - o(n)$	[Find, Golovnev, Hirsch, Kulikov 2016]
U_2	$3n - O(1)$	[Schnorr 1976]
U_2	$4n - O(1)$	[Zwick 1991]
U_2	$4.5n - O(1)$	[Lachish, Raz 2001]
U_2	$5n - o(n)$	[Iwama, Morizumi 2002]

Restricted circuit classes

- The good news: We can prove super-linear lower bounds for restricted circuits
- In the most famous theorems, the "hard function" is in P

Hard function	Model	Size lower bound	Reference
Parity	Depth-d AC^0	$\exp(n^{\Omega(1/d)})$	[Ajtai 1983] [Furst, Saxe, Sipser 1984] [Yao 1985] [Håstad 1986]
Majority	Depth-d $\mathrm{AC}^0[\oplus]$	$\exp(n^{\Omega(1/d)})$	[Razborov 1987] [Smolensky 1987] [Oliveira, Santhanam, Srinivasan 2019]
Andreev's function	De Morgan formulas	$\widetilde{\Omega}(n^3)$	[Andreev 1987] [Impagliazzo, Nisan 1993] [Paterson, Zwick 1993] [Håstad 1998] [Tal 2014]
Parity	Depth-d TC^0	$n^{1+2^{-\Theta(d)}}$	[Impagliazzo, Paturi, Saks 1997]

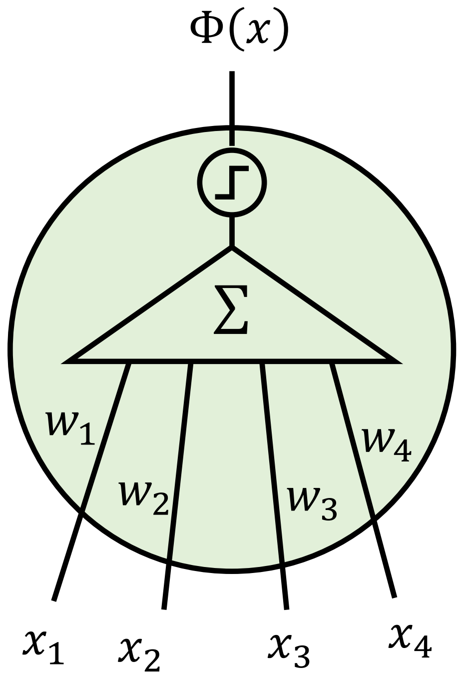


Hard functions in uniform AC^0

- Can we prove super-linear lower bounds where the hard function is in uniform AC^0?
  - Recall: AC^0 = constant-depth poly-size unbounded-fan-in circuits using AND/OR/NOT gates
  - Can we prove circuit lower bounds for "hyperexplicit" functions?

Model	Size lower bound	Hard function's AC^0 parameters	Reference
Depth-d AC^0	$\exp(n^{\Omega(1/d)})$	Depth $d+1$ Size $O(n)$	[Sipser 1983] [Yao 1985] [Håstad 1987]
Depth-d $\mathrm{AC}^0[\oplus]$	$n^{(\log n)^{\Omega(1/d)}}$	Depth $d+1$ Size $O(n)$	[O'Donnell, Wimmer 2007] [Amano 2009] [Limaye, Sreenivasaiah, Srinivasan, Tripathi, Venkitesh 2021]
Depth-d $\mathrm{AC}^0[\oplus]$	n^k	Depth d Size $n^{O(k)}$	[Limaye, Sreenivasaiah, Srinivasan, Tripathi, Venkitesh 2021] [Limaye, Srinivasan, Tripathi 2019]
De Morgan formulas	$n^{3-o(1)}$	Depth 4 Size $O(n^3)$	[Filmus, Meir, Tal 2021]
Depth-d TC^0	???	???	???

TC^0 circuit model



- Each gate computes a linear threshold function (LTF):

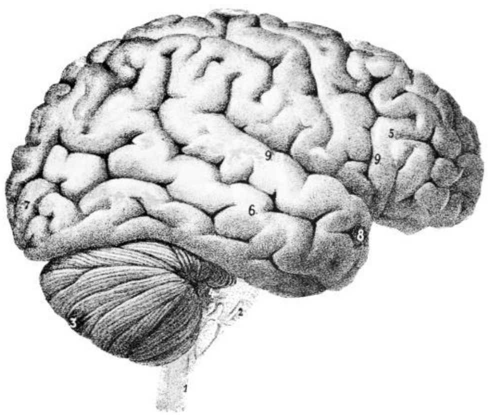
  $\Phi(x) = 1 \Leftrightarrow \sum_{i=1}^{n} w_i x_i \geq \theta$ where $w \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$

- AND, OR, MAJORITY, …

- Boolean analogue of a neural network

- Size = number of wires

- TC^0: Depth = $O(1)$, size = $\mathrm{poly}(n)$



Super-linear lower bounds for $\mathbf{TC}^0$ circuits

- Let $n, d \in \mathbb{N}$ with $d = o(\log\log n)$

Theorem [Impagliazzo, Paturi, Saks 1997]: $\exists \gamma = 2^{-\Theta(d)}$ such that the parity function on n bits cannot be computed by depth-d LTF circuits with $n^{1+\gamma}$ wires

Our Theorem: $\exists \gamma = 2^{-\Theta(d)}$ and $\exists F: \{0,1\}^n \to \{0,1\}$ such that

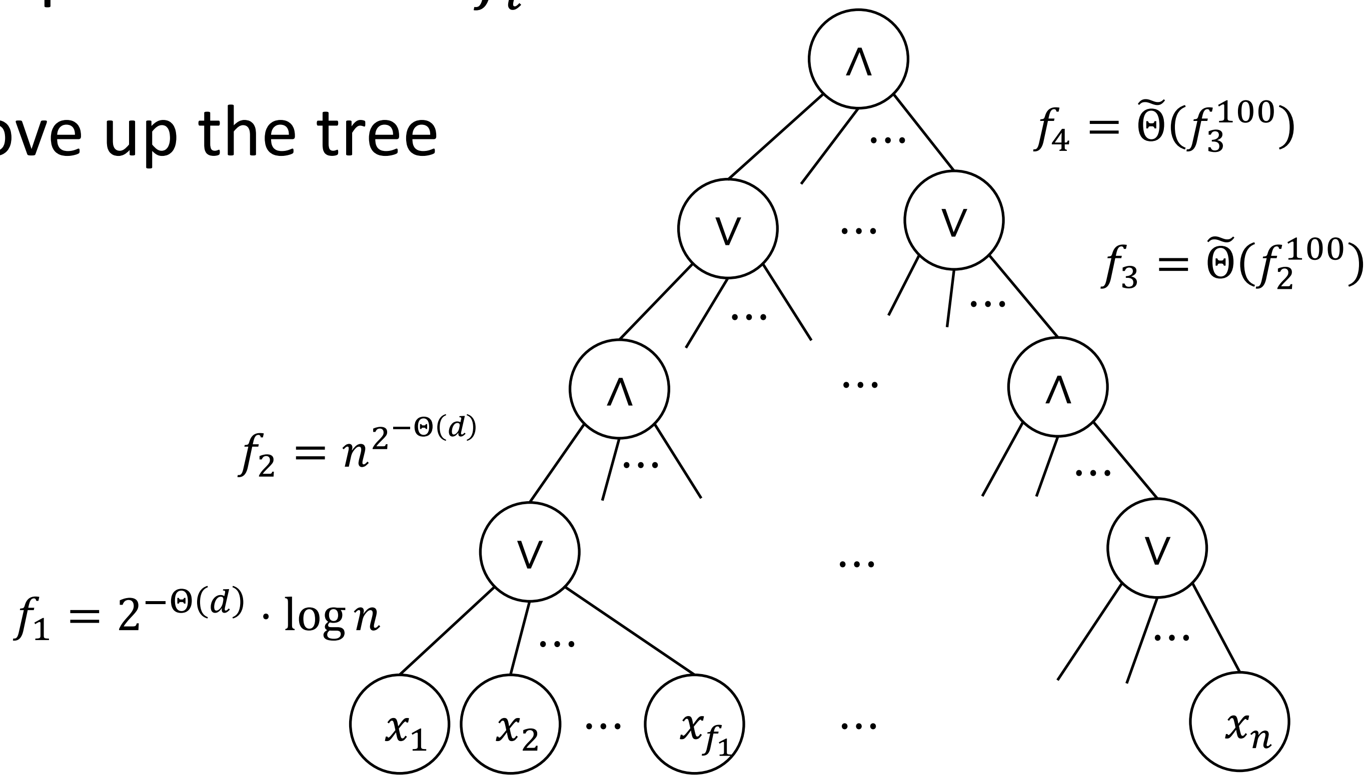
- F can be computed by an explicit depth-$(d+1)$ AC^0 circuit with $O(n)$ wires
- F cannot be computed by depth-d LTF circuits with $n^{1+\gamma}$ wires

- Many prior lower bounds for simulating AC^0 by subclasses of LTF circuits such as monotone LTF circuits, $\mathrm{MAJ} \circ \mathrm{LTF}$ circuits, etc.

Our hard function F

- F is a monotone read-once AC^0 formula (an AND-OR tree)
- Every gate at distance i from the input has fan-in f_i
- The fan-ins grow rapidly as we move up the tree

  $(f_1 \ll f_2 \ll \cdots \ll f_{d+1})$



Our lower bound approach

- We build on the proof of the Average-Case Depth Hierarchy Theorem for AC^0 [Håstad, Rossman, Servedio, Tan 2017]
- A projection is a map $\pi: \{x_1, \ldots, x_n\} \to \{0, 1, y_1, \ldots, y_m\}$ where x_i, y_j are variables
  - Traditional restrictions are the special case $\pi(x_i) \in \{0, 1, x_i\}$
- Following [HRST17], we show that under a suitable random projection, an LTF circuit becomes near-trivial while our AND-OR tree F maintains structure
- Our analysis of LTF circuits builds on prior analyses of LTF circuits under random restrictions, especially [Chen, Santhanam, and Srinivasan 2018]

Summary

- We construct an explicit $F \in \mathrm{AC}^0$ that cannot be computed by depth-d LTF circuits with $n^{1+\gamma}$ wires
- Our result essentially matches the best lower bounds for depth-d LTF circuits ($d > 2$), with one bonus feature: now the hard function is in AC^0
- Thanks for listening! Questions?