Depth-d Threshold Circuits vs. Depth-$(d+1)$ AND-OR Trees

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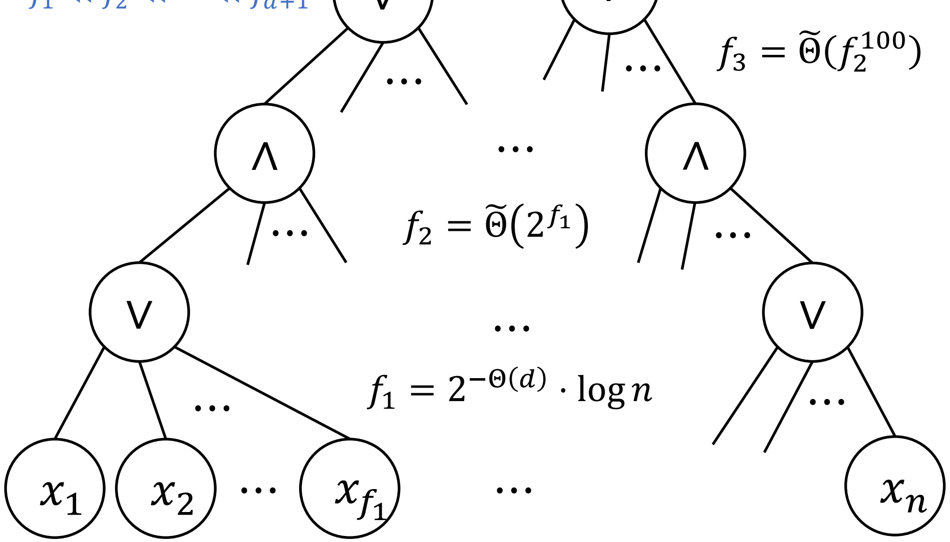
- Let n be sufficiently large, let $d \le 0.05 \log\log n$, and let $\gamma = 2^{-10\cdot d}$
- **Main Theorem:** $\exists F: \{0,1\}^n \to \{0,1\}$ such that
  - F can be computed by an explicit depth-$(d+1)$ AC^0 circuit with $O(n)$ wires…
  - …but F cannot be computed by any depth-d threshold circuit with $n^{1+\gamma}$ wires

Hard function F: AND-OR tree

- f_i = fan-in of gates at distance i from inputs
- $f_1 \ll f_2 \ll \cdots \ll f_{d+1}$



Background

- In a "threshold circuit," each gate computes a linear threshold function: $\Phi(x) = 1 \Leftrightarrow \sum_{i=1}^{n} w_i \cdot x_i \ge \theta$ where $w \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$
- TC^0 = constant-depth poly-size threshold circuits
- Open problem: Prove $\mathrm{NEXP} \not\subseteq \mathrm{TC}^0$ (progress toward $\mathrm{P} \neq \mathrm{NP}$)
- **Theorem** (Impagliazzo, Paturi, Saks 1997): $\exists \gamma = 2^{-\Theta(d)}$ s.t. parity cannot be computed by depth-d threshold circuits with $n^{1+\gamma}$ wires
- Prior lower bounds for simulating AC^0 by subclasses of threshold circuits (monotone threshold circuits, $\mathrm{MAJ} \circ \mathrm{LTF}$ circuits, etc.)

Our lower bound approach

- A projection is a map $\pi: \{x_1, \dots, x_m\} \to \{0, 1, y_1, \dots, y_m\}$ where $x_1, \dots, x_n, y_1, \dots, y_m$ are variables. Restriction: $\pi(x_i) \in \{0, 1, x_i\}$
- Building on work by Håstad, Rossman, Servedio, and Tan (2017), we design a random projection π s.t. for a suitable distribution σ:

(1) W.h.p. over π, the projected AND-OR tree is balanced: $\Pr_\sigma[F|_\pi(\sigma) = 1] = 1/2 \pm o(1)$

(2) $\forall$ threshold circuit f with $n^{1+\gamma}$ wires, w.h.p. over π, the projected circuit is biased: $\exists b$ s.t. $\Pr_\sigma[f|_\pi(\sigma) = b] = 1 - o(1)$

- Proof of (2) builds on prior analyses of threshold circuits under random restrictions, especially work on average-case lower bounds by Chen, Santhanam, and Srinivasan (2018)