Typically-Correct Derandomization for Small Time and Space

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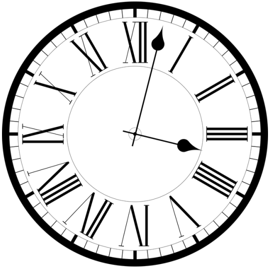
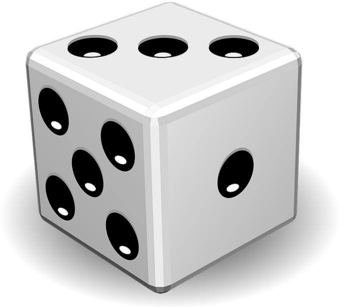
HUJI CS Theory Seminar

[1] Supported by the NSF GRFP under Grant No. DGE1610403.

Time, space, and randomness





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  - [Saks, Zhou '95]: Space $\Theta(\log^{1.5} n)$

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- Naïve derandomization: Run $A(x, x)$
- Might fail on all x because of correlations between input, coins

Prior techniques for dealing with correlations

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   - [Kinne, van Melkebeek, Shaltiel '12]: Multiparty communication protocols, $\mathbf{BPAC}^0$ with symmetric gates

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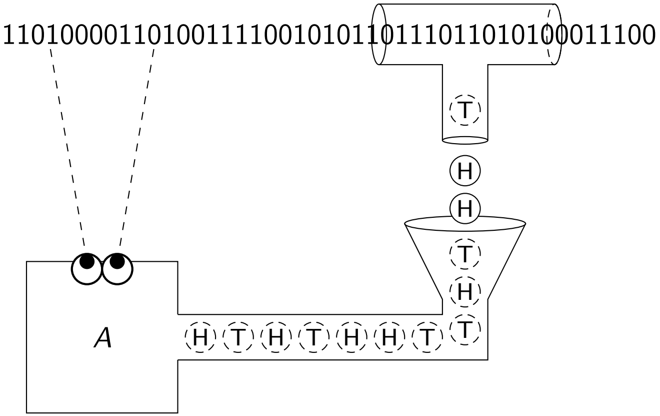

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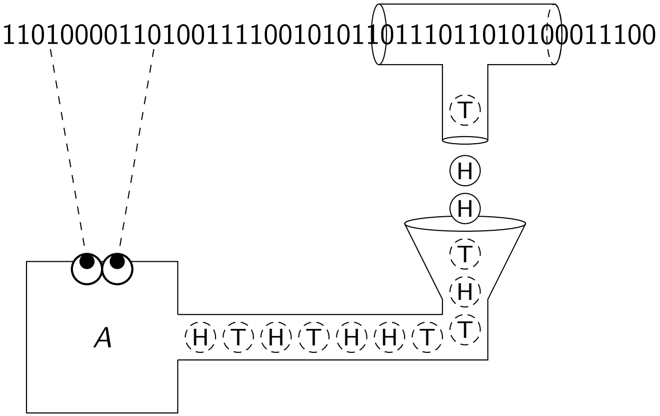
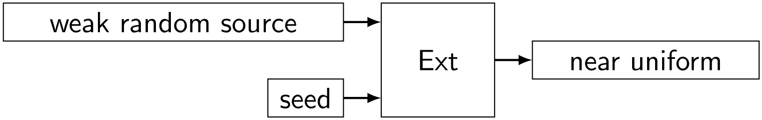


- (Additional ideas needed to make this work...)

Randomness extractors



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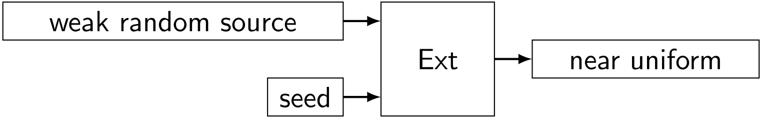

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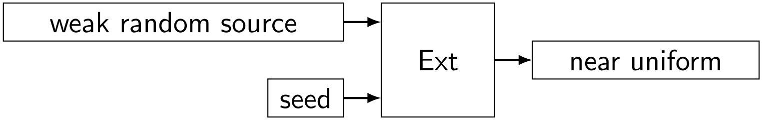

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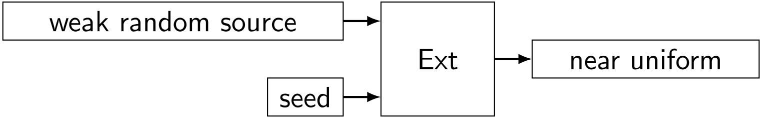

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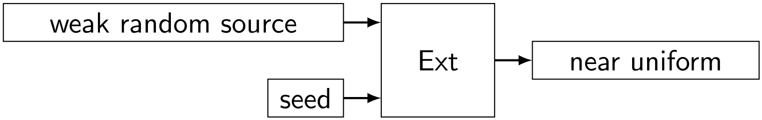

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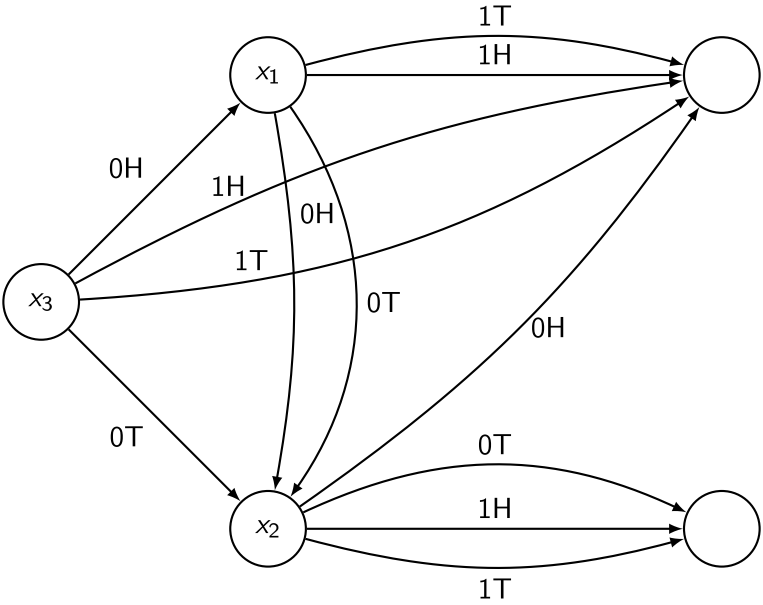
- Think $s \approx k$ and $d \approx \log(\ell/\varepsilon)$.

Randomized branching programs



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- $\mathcal{P}(v; x, y)$ = the terminal vertex reached if you start from vertex v, read input $x \in \{0,1\}^n$, use random bits $y \in \{0,1\}^T$

Nisan's generator

- **Theorem** (Nisan '92): There is a pseudorandom generator

$$\text{NisGen} : \{0,1\}^s \to \{0,1\}^T \tag{1}$$

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  Caveat: Sampling error is large for tiny fraction of x values

Restriction of a program

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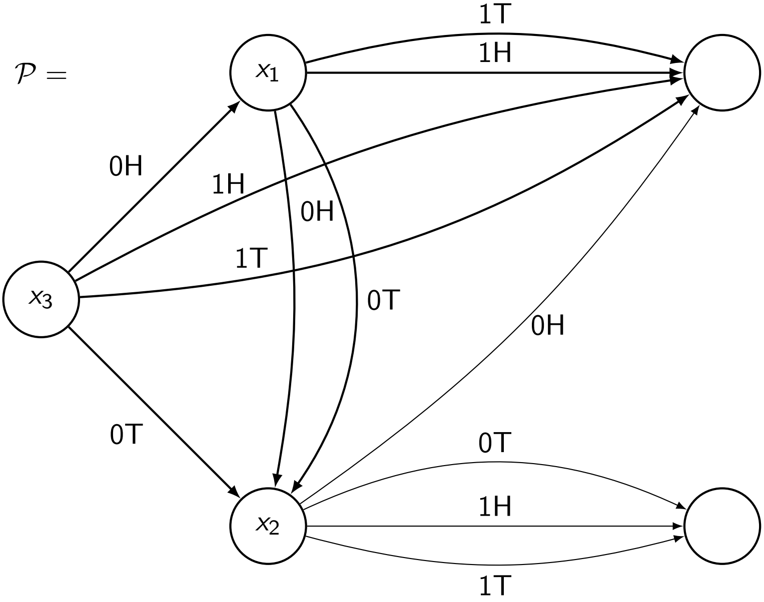

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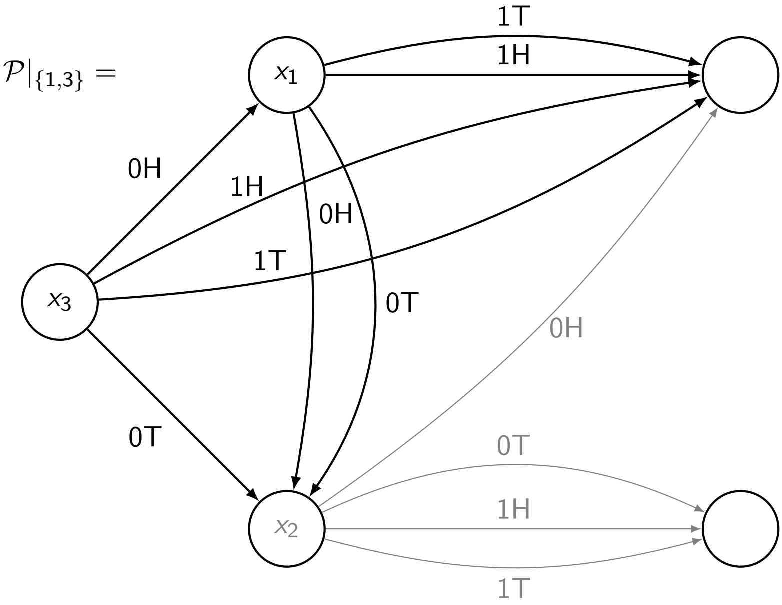

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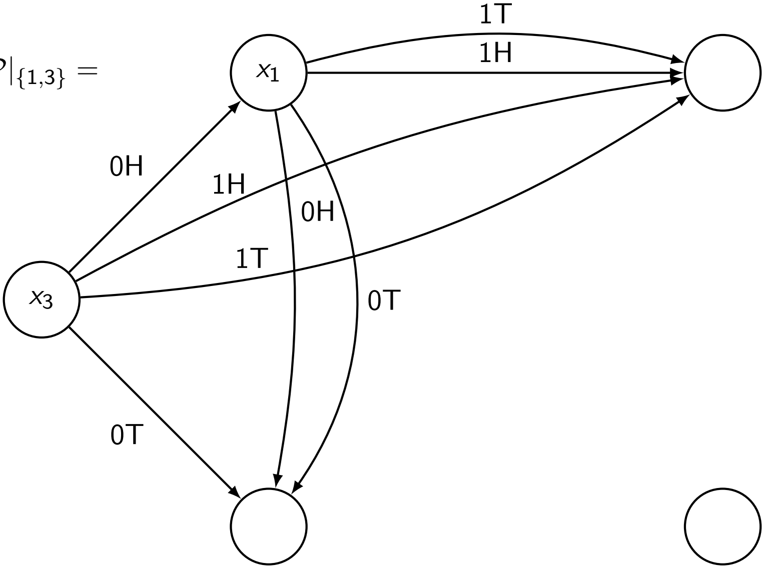

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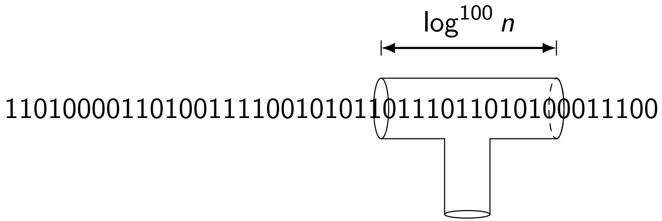

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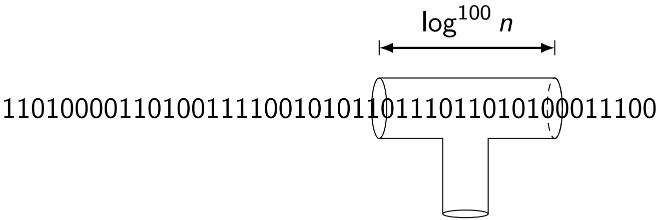

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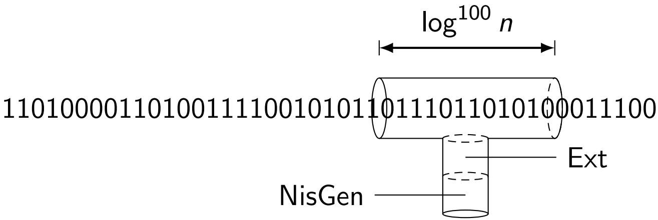

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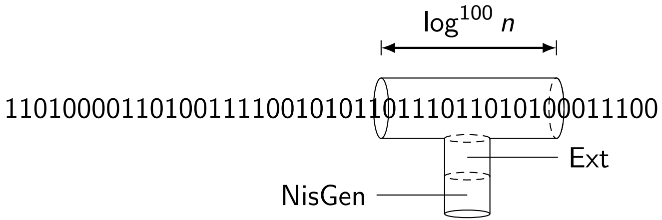

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  - $(\#\text{ bad } x \leq 2^{k+1}|V|)$

Correctness proof sketch

True algorithm

1. Pick random $y \in \{0,1\}^{O(\log n)}$
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Correctness proof sketch

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Second hybrid distribution

1. Initialize $v := v_0$
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The Nisan-Zuckerman generator

- **Theorem** (Nisan, Zuckerman '96): For every constant c, there is a pseudorandom generator

$$\mathsf{NZGen} : \{0,1\}^d \to \{0,1\}^{\log^c n}$$

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  - Roughly: Derandomization with $< n$ bits of advice $\implies$ typically-correct derandomization with no advice

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- **Proof of correctness**: # bad a bounded by

$$\underbrace{\left(2^{O(\log^2 n)}\right)}_{\#\text{ bad } a \text{ for each } x} \cdot \underbrace{(2^n)}_{\#\, x} < 2^{|a|}$$

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- **Theorem**: If $L \in \mathbf{BPL}$ admits a $\mathbf{DSPACE}(\log n)$ algorithm A that fails on ε-fraction of inputs, then

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- (Detail: Make advice algorithm "zero-error" using $\mathbf{RL} \subseteq \mathbf{SC}$ trick)

Derandomizing quasilinear-time, log-space with advice

- **Corollary**: For every constant c,

$$\mathbf{BPTISP}(\widetilde{O}(n), \log n) \subseteq \mathbf{L}/(n - \log^c n).$$

Sublinear advice

- $\mathbf{BPTISP}_{\mathrm{TM}}(T, S)$: Time-$T$ space-S multitape Turing machines
- **Theorem**: For every constant c,

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Beyond quasilinear time

- **Theorem**:

$$\mathbf{BPTISP}_{\mathsf{TM}}(n^{1.99}, \log n) \subseteq \mathbf{L}/(n - n^{\Omega(1)}).$$

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- Thanks! Questions?