Typically-Correct Derandomization for Small Time and Space

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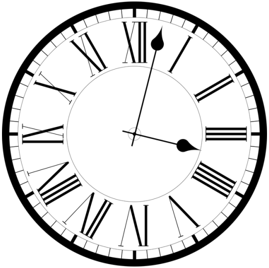
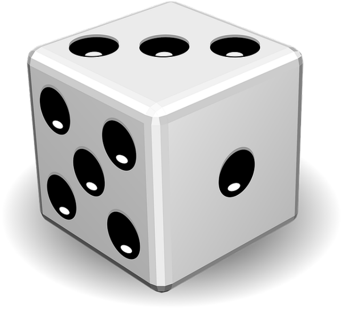
July 18
CCC 2019

[1] Supported by the NSF GRFP under Grant No. DGE1610403 and by a Harrington fellowship from UT Austin

Time, space, and randomness





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  - [Saks, Zhou '95]: Space $\Theta(\log^{1.5} n)$

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- Naïve derandomization: Run $A(x, x)$
- Might fail on all x because of correlations between input, coins

Prior techniques for dealing with correlations

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   - [Kinne, van Melkebeek, Shaltiel '12]: Multiparty communication protocols, $\mathbf{BPAC}^0$ with symmetric gates

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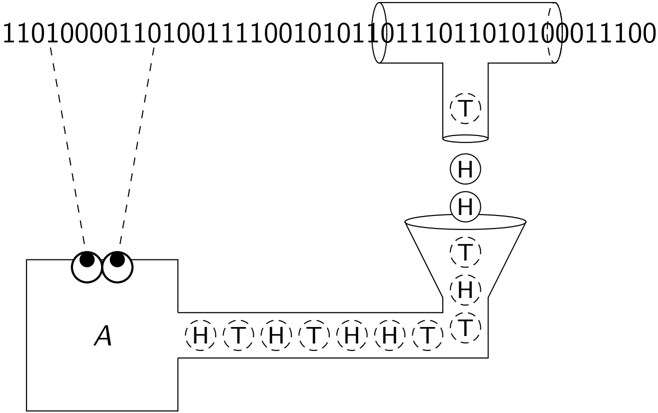

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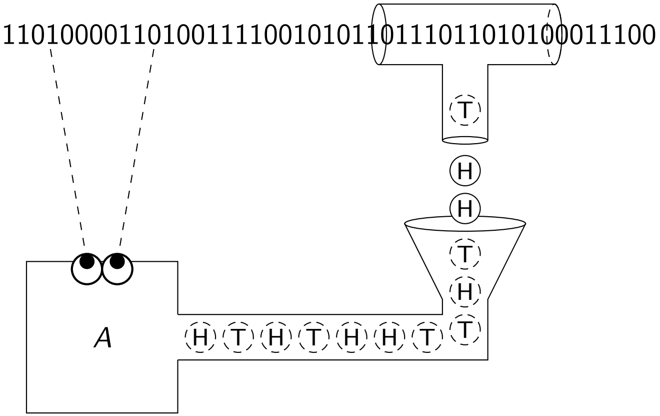

- (Additional ideas needed to make this work...)

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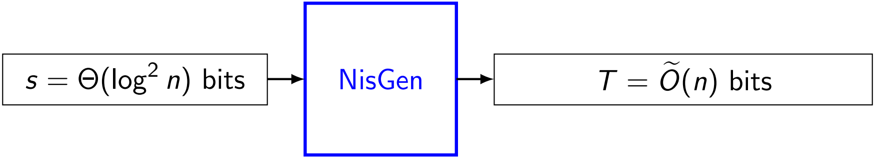
Tool 1: Nisan's Pseudorandom Generator



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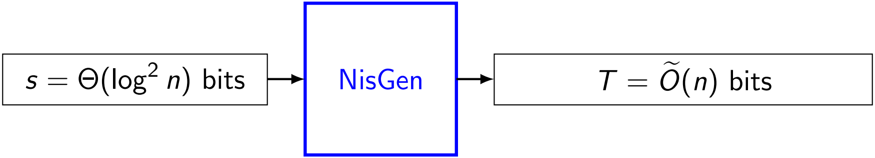

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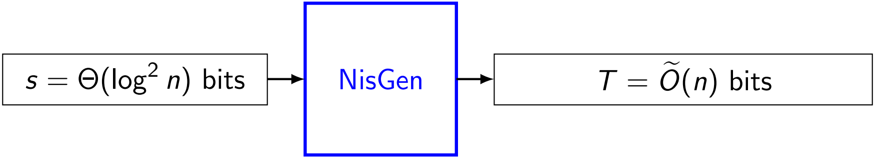

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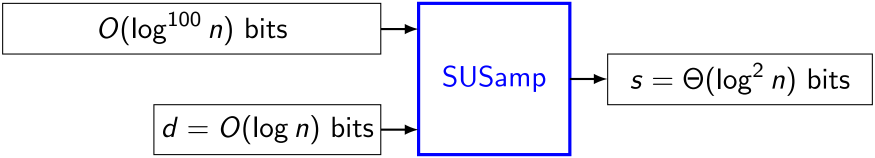
- ...given two-way access to seed

Tool 2: Shaltiel-Umans Averaging Sampler



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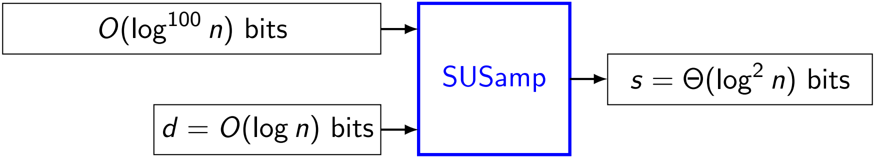

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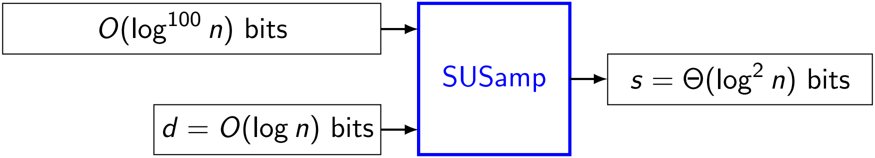

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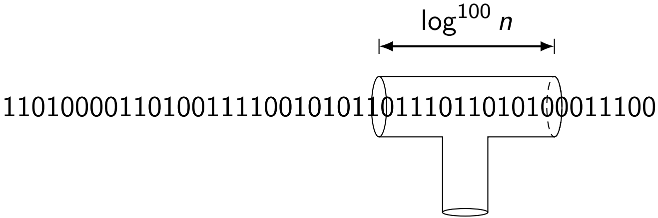

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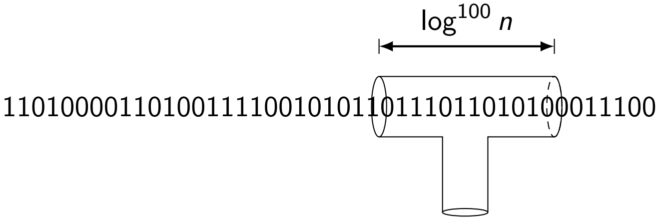

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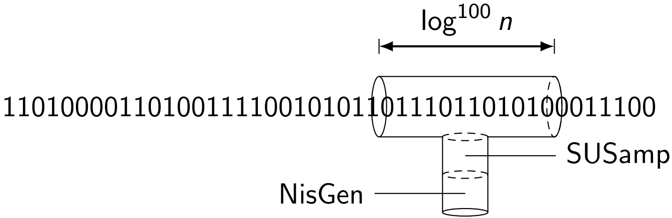

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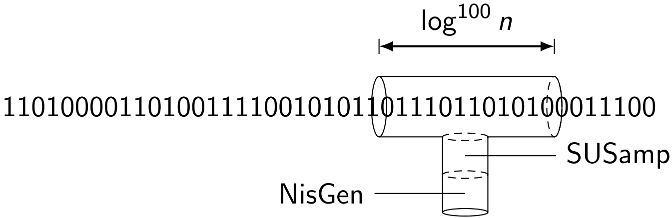

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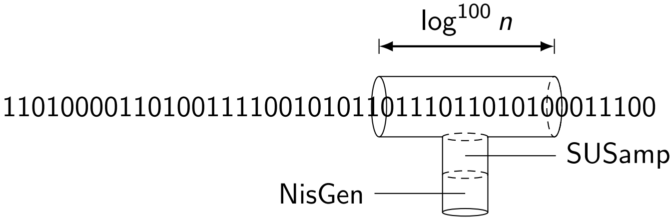

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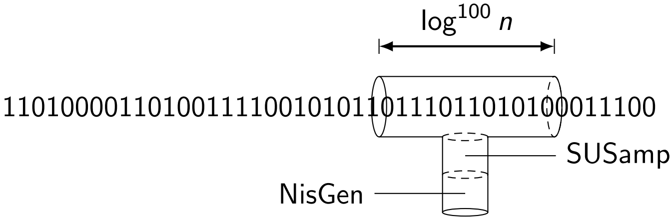

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3. Accept if v is an accepting configuration, else reject

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- Randomness polylog n (one-way access!)

Correctness proof sketch

Our algorithm

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Correctness proof sketch

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- Let $F(y')$ = final configuration when running $A|_{[n]\setminus I}(x, \text{NisGen}(y'))$ from v

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3. Run $A\|_{[n]\setminus I}(x, z)$ from v	3. Run $A\|_{[n]\setminus I}(x, z)$ from v
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- Let $F(y') =$ final configuration when running $A|_{[n]\setminus I}(x, \mathrm{NisGen}(y'))$ from v
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Correctness proof sketch

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Correctness proof sketch (3)

Second hybrid distribution

1. Repeat polylog(n) times:
   1.1 Pick random $I \subseteq [n]$
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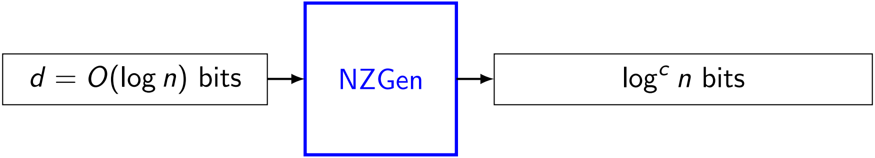
- After polylog(n) phases, reach halting configuration w.h.p. $\square$

Tool 3: The Nisan-Zuckerman PRG



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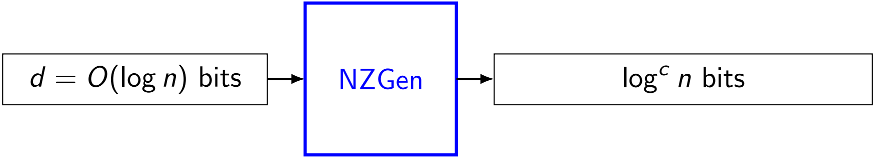

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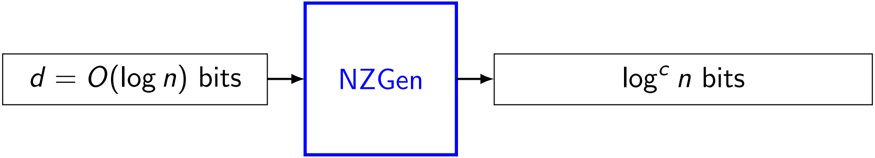

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- Thanks! Questions?