Universal Bell Correlations Do Not Exist

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We prove that there is no finite-alphabet nonlocal box that generates exactly those correlations that can be generated using a maximally entangled pair of qubits. More generally, we prove that if some finite-alphabet nonlocal box is strong enough to simulate arbitrary local projective measurements of a maximally entangled pair of qubits, then that nonlocal box cannot itself be simulated using any finite amount of entanglement. We also give a quantitative version of this theorem for approximate simulations, along with a corresponding positive result.

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A classic goal of quantum information research is to understand the power of quantum nonlocality: Which nonlocal tasks can be performed using quantum entanglement? For example, quantum entanglement is helpful for playing certain nonlocal games, but it cannot be used to achieve faster-than-light communication. The notion of a correlation box serves as a conceptual tool for reasoning about this question. A correlation box is a hypothetical randomized "channel" through which two separated parties, Alice and Bob, can interact. Mathematically, a correlation box is a function $C: X \times Y \to \Delta(A \times B)$, where X, Y, A, B are countable (finite or countably infinite) alphabets and $\Delta(A \times B)$ denotes the set of all probability distributions over $A \times B$. We imagine that Alice chooses $x \in X$ and Bob chooses $y \in Y$. A sample (a, b) is drawn from $C(x, y)$, and Alice is given a and Bob is given b. We will abuse notation and write $C: X \times Y \to A \times B$.

The canonical example [1,2] is the Popescu-Rohrlich (PR) box $C_{\mathrm{PR}}: \{0,1\} \times \{0,1\} \to \{0,1\} \times \{0,1\}$, defined by

$$C_{\mathrm{PR}}(x, y) = \begin{cases} (0, xy) & \text{with probability} \quad 1/2, \\ (1, 1 - xy) & \text{with probability} \quad 1/2. \end{cases} \qquad (1)$$

PR boxes cannot be used to communicate, since the marginal distributions of a and b are uniform regardless of x and y. However, C_{PR} is a nonlocal box: given access to a PR box, Alice and Bob can perform tasks that would be impossible if they were isolated in a classical world. The standard example is winning the Clauser-Horne-Shimony-Holt (CHSH) game [3] with certainty. As usual, we model the classical scenario by assuming that each player's behavior is a function of his or her private input and a random variable shared between the two players.

For any correlation box C, rather than analyzing the capabilities of two parties with access to C, we can instead analyze the problem of simulating C. That is, Alice is given $x \in X$ and Bob is given $y \in Y$. Alice is supposed to output $a \in A$ and Bob is supposed to output $b \in B$ such that (a, b) has the distribution $C(x, y)$. Let $\mathbf{Q}$ be the class of all correlation boxes that can be simulated if Alice and Bob have unlimited shared randomness and an arbitrary but finite amount of entanglement. The question at the beginning of this Letter can now be sharpened: Which correlation boxes are in $\mathbf{Q}$? For example, the Tsirelson bound [4] implies that $C_{\mathrm{PR}} \notin \mathbf{Q}$.

Let $\mathbf{B}$ be the class of all correlation boxes that can be simulated if Alice and Bob have unlimited shared randomness, each holds one of a pair of maximally entangled qubits, and they are only allowed to make projective measurements. Clearly, $\mathbf{B} \subseteq \mathbf{Q}$; understanding $\mathbf{B}$ is a good first step toward understanding $\mathbf{Q}$. Bell [5] famously showed that there are correlation boxes in $\mathbf{B}$ that cannot be simulated using only shared randomness.

A long line of work [6–13] investigated the problem of simulating Bell correlations using classical communication, culminating in a protocol by Toner and Bacon [13] for simulating any correlation box in $\mathbf{B}$ using shared randomness and a single classical bit of one-way communication. Cerf *et al.* [14] improved on the Toner-Bacon theorem by showing that instead of a bit of communication, it suffices to have a single PR box. (Taking a cue from quantum mechanics, we think of correlation boxes as "single use only.")

In general, if every correlation box in $\mathbf{B}$ can be simulated using a particular correlation box C, then C can be interpreted as an upper bound on the power of $\mathbf{B}$. Part of what makes the result by Cerf *et al.* so appealing is that C_{PR} has finite alphabets, making it an extremely explicit upper bound on the power of $\mathbf{B}$: a PR box is a "discrete" device. On the other hand, a "defect" of the result by Cerf *et al.* is that $C_{\mathrm{PR}} \notin \mathbf{B}$, and hence C_{PR} is a loose upper bound. Local projective measurements of a Bell pair can be simulated using a PR box, but not vice versa.

It is natural, therefore, to hope to construct some finite-alphabet correlation box C_* such that every correlation box in $\mathbf{B}$ can be simulated using C_*, and C_* is in $\mathbf{B}$. Such a correlation box C_* would exactly characterize $\mathbf{B}$, greatly clarifying the power of quantum nonlocality. Unfortunately, our main result is that no such correlation box C_* exists.

Actually, the situation is even worse in several respects. The simulations we have discussed so far are one-query reductions. In general, if C_1 and C_2 are correlation boxes, a k-query reduction from C_1 to C_2 is a protocol for simulating C_1 in which Alice and Bob have unlimited shared randomness and k copies of C_2 that they use in a prespecified order. We say that C_1 reduces to C_2 if there is a k-query reduction from C_1 to C_2 for some finite k. We prove that C_* does not exist even if we allow countably infinite input alphabets, we allow an arbitrary finite amount of entanglement when simulating C_*, we consider general reductions, and we only try to simulate correlation boxes in $\mathbf{B}$ with binary alphabets.

Theorem 1: Suppose $C_* \in \mathbf{Q}$ has countable input alphabets and finite output alphabets. Then, there is some correlation box $C: \{0,1\} \times \{0,1\} \to \{0,1\} \times \{0,1\}$ such that $C \in \mathbf{B}$, but C does not reduce to C_*.

Adopting terminology from computational complexity theory, if C is a correlation box and $\mathbf{C}$ is a class of correlation boxes, we say that C is $\mathbf{C}$-complete if $C \in \mathbf{C}$ and every correlation box in $\mathbf{C}$ reduces to C.

Corollary 1: There does not exist a finite-alphabet $\mathbf{B}$-complete correlation box.

Corollary 2: There does not exist a finite-alphabet $\mathbf{Q}$-complete correlation box.

Our results can be thought of as "bad news" for the project of understanding the power of quantum nonlocality. In this respect, our results are in the same spirit as previous work showing that certain information-theoretic conditions fail to exactly characterize quantum correlations [15,16]. Our results should also be considered in the context of the body of research [17–23] investigating the power of correlation boxes in their own right, apart from quantum entanglement. In particular, Barrett and Pironio [17] gave a reduction from any nonsignaling correlation box with binary output alphabets to C_{PR}. Our result shows that there is no corresponding phenomenon for $\mathbf{B}$. Furthermore, Dupuis *et al.* [19] showed that no finite-alphabet correlation box is complete for the class of nonsignaling correlation boxes; our result is analogous.

We now sketch a proof of a weaker version of Theorem 1. We will consider a certain class of nonlocal games, parametrized by a real value p. Lawson, Linden, and Popescu [24] showed that the optimal entangled success probability for these games depends nonlinearly on p. On the other hand, if C_* is a finite-alphabet correlation box, the optimal success probability of a strategy using shared randomness and a single query to C_* can be shown to be some piecewise-linear function of p. Hence, C_* is not $\mathbf{Q}$-complete with respect to one-query reductions. A more careful analysis will justify the stronger claim expressed by Theorem 1. We remark that our argument is similar in spirit to Bell's original proof [5] of his namesake theorem.

We also give a quantitative version of our result for approximate simulations. An ε-error reduction is defined like an ordinary reduction except that we allow ε total variation error.

Theorem 2: Suppose $C_*: X \times Y \to A \times B$ is a finite-alphabet correlation box in $\mathbf{Q}$. Then, there is some correlation box $C: \{0,1\} \times \{0,1\} \to \{0,1\} \times \{0,1\}$ in $\mathbf{B}$ such that for every k, if there is a k-query ε-error reduction from C to C_*, then

$$k^4 \times (2|X|)^{4|A|^k} \times (2|Y|)^{4|B|^k} \geq \Omega(1/\varepsilon). \quad (2)$$

Conversely, for any $\varepsilon > 0$, we give a simple construction of $C_*: X \times Y \to \{1,-1\} \times \{1,-1\}$ with $|X| = |Y| \leq O(1/\varepsilon^2)$ such that $C_* \in \mathbf{B}$ and every correlation box in $\mathbf{B}$ reduces to C_* via a one-query ε-error reduction. Note that for $|A| = |B| = 2$, $k = 1$, Theorem 2 implies that $|X| \times |Y|$ must be at least $1/\varepsilon^{\Omega(1)}$. On the other hand, when $|A|$, $|B|$, k are large, Theorem 2 might be very far from tight.

Proofs of negative results.—For $0 \leq p, q \leq 1$, the biased CHSH game CHSH$[p,q]$ is defined as follows [24]. The referee picks $x, y \in \{0,1\}$ independently at random, with $\Pr[x=1] = p$, $\Pr[y=1] = q$. Alice gets x and Bob gets y. Alice outputs $a \in \{0,1\}$ and Bob outputs $b \in \{0,1\}$. They win if $a + b = xy \pmod 2$. Alice and Bob know p and q. The standard CHSH game [3] is the case $p = q = \frac{1}{2}$.

We can think of a correlation box $C: \{0,1\} \times \{0,1\} \to \{0,1\} \times \{0,1\}$ as a strategy for the biased CHSH game. The probability that C wins CHSH$[p,q]$ is the probability that $a + b = xy \pmod 2$, where $(a,b) = C(x,y)$ and the probability is over both the internal randomness of C and the inputs x, y. [The inputs (x,y) are independent of the internal randomness of C.] Lawson *et al.* computed the optimal success probability for entangled strategies.

Lemma 1 [24]: If $\frac{1}{2} \leq q \leq 1/2p \leq 1$, then there exists $C_{p,q} \in \mathbf{B}$ that wins CHSH$[p,q]$ with probability $\frac{1}{2} + \frac{1}{2}\sqrt{2}\sqrt{q^2 + (1-q)^2}\sqrt{p^2 + (1-p)^2}$. Furthermore, no correlation box in $\mathbf{Q}$ achieves higher win probability.

Throughout the rest of the Letter, we define

$$\omega(p) = \frac{1}{2} + \frac{1}{2}\sqrt{p^2 + (1-p)^2}, \quad (3)$$

which is the bound of Lemma 1 for $q = 1/2$. We now prove that playing CHSH$[p,1/2]$ near optimally using some $C_* \in \mathbf{Q}$ requires approximating ω by a small set of linear functions (i.e., polynomials in p of degree 1). Roughly, the idea is that there are only so many things one can do with C_*, and each of them gives rise to a strategy for the biased CHSH game with a success probability that depends linearly on p.

Lemma 2. Suppose $C_*: X \times Y \to A \times B$ is in $\mathbf{Q}$ and $k \in \mathbb{N}$. For $p \in [1/2, 1]$, let $C_{p,1/2}$ be the box of Lemma 1. There is a set $L_{C_*,k}$ of linear functions $\mathbb{R} \to \mathbb{R}$ such that the following conditions hold.

(1) For every $p \in [1/2, 1]$ and every $\varepsilon > 0$, if there exists a k-query ε-error reduction from $C_{p,1/2}$ to C_*, then there exists $\ell \in L_{C_*,k}$ such that $|\ell(p) - \omega(p)| \le \varepsilon$.

(2) If X, Y, A, B are all finite, then $|L_{C_*,k}|$ is at most $(2|X|)^{2|A|^k} \times (2|Y|)^{2|B|^k}$. If X, Y are countable and A, B are finite, then $L_{C_*,k}$ is countable.

Proof.—A deterministic k-query C_* protocol is a protocol that uses k copies of C_* in a deterministic way. (The output of such a protocol is random, but only because of the internal randomness of C_*.) For each such protocol Π, let $\ell_\Pi(p)$ be the probability that Π wins CHSH$[p, 1/2]$. Then, ℓ_Π is a linear function, since

$$\ell_\Pi(p) = \frac{1-p}{2} P_{00} + \frac{1-p}{2} P_{01} + \frac{p}{2} P_{10} + \frac{p}{2} P_{11}, \quad (4)$$

where P_{xy} is the probability that $a + b = xy \pmod 2$ where $(a, b) = \Pi(x, y)$. Let $L_{C_*,k}$ be the set of all ℓ_Π.

To prove the first item, let Λ be a k-query ε-error reduction from $C_{p,1/2}$ to C_*. We can think of Λ as a distribution over deterministic C_*-protocols Π. By the correctness of the reduction, $|E_{\Pi\sim\Lambda}[\ell_\Pi(p)] - \omega(p)| \le \varepsilon$. The best case is at least as good as the average case, so there is a deterministic protocol Π_* such that $\ell_{\Pi_*}(p) \ge \omega(p) - \varepsilon$. Since $C_* \in \mathbf{Q}$ and $\mathbf{Q}$ is closed under reductions, Π_* implements a correlation box in $\mathbf{Q}$. Therefore, by the optimality clause of Lemma 1, $\ell_{\Pi_*}(p) \le \omega(p)$, and hence $|\ell_{\Pi_*}(p) - \omega(p)| \le \varepsilon$.

To prove the second item, we bound the number of deterministic k-query C_* protocols. Such a protocol can be specified by the following data.

(i) Functions $q_i: \{0, 1\} \times A^{i-1} \to X$ for each $1 \le i \le k$, telling the ith query that Alice makes as a function of her input and the query responses she has seen so far, and corresponding functions $r_i: \{0, 1\} \times B^{i-1} \to Y$ for Bob.

(ii) A function $s: \{0, 1\} \times A^k \to \{0, 1\}$, telling the output Alice gives as a function of her input and all query responses, and a corresponding function $t: \{0, 1\} \times B^k \to \{0, 1\}$ for Bob.

If X, Y are countable and A, B are finite, then there are only countably many possibilities for each of these functions, so there are countably many such protocols. Suppose now that X, Y, A, B are all finite and $|A|, |B| \ge 2$. The number of possibilities for q_i is $|X|^{2|A|^{i-1}}$, and similarly for r_i. The number of possibilities for s is $2^{2|A|^k}$, and similarly for t. Therefore, $|L_{C_*,k}|$ is bounded by

$$\left(\prod_{i=1}^{k} |X|^{2|A|^{i-1}}\right)\left(\prod_{i=1}^{k} |Y|^{2|B|^{i-1}}\right) \times 2^{2|A|^k} \times 2^{2|B|^k} \quad (5)$$

$$= |X|^{2\sum_i |A|^{i-1}} \times |Y|^{2\sum_i |B|^{i-1}} \times 2^{2|A|^k} \times 2^{2|B|^k} \quad (6)$$

$$\le |X|^{2|A|^k} \times |Y|^{2|B|^k} \times 2^{2|A|^k} \times 2^{2|B|^k} \quad (7)$$

$$= (2|X|)^{2|A|^k} \times (2|Y|)^{2|B|^k}. \quad (8)$$

Finally, if A is a singleton set, the step above where we bounded $\sum_i |A|^{i-1}$ by $|A|^k$ was not valid, but in this case $q_1, \ldots, q_k$ do not need to be specified anyway, so the final bound still holds, and similarly if B is a singleton set. ■

Next, we show that ω cannot be well approximated by a small set of linear functions. To prove Theorem 1, the following trivial fact suffices.

Lemma 3: Suppose L is a countable set of linear functions $\mathbb{R} \to \mathbb{R}$. Then, there is some $p \in [1/2, 1]$ such that for every $\ell \in L$, $\ell(p) \ne \omega(p)$.

Proof.—Suppose $\ell(p) = \omega(p)$, where $\ell \in L$. Rearranging,

$$p^2 + (1-p)^2 = r(p)^2, \quad (9)$$

where $r(p)$ is another linear function. The quadratic expression on the left hand side of Eq. (9) is not a square (e.g., the discriminant is $-4 \ne 0$). Therefore, Eq. (9) is a nondegenerate quadratic equation, so it has at most two solutions p. So ℓ intersects ω at most twice, and hence L intersects ω in countably many places. ■

Proof of Theorem 1.—Fix $C_*: X \times Y \to A \times B$, where X, Y are countable, A, B are finite, and $C_* \in \mathbf{Q}$. We will show that there is some choice of p so that there is no reduction from $C_{p,1/2}$ to C_*; since $C_{p,1/2} \in \mathbf{B}$ and $C_{p,1/2}$ has binary alphabets, this will complete the proof.

For each $k \in \mathbb{N}$, let $L_{C_*,k}$ be the set of linear functions given by Lemma 2. The alphabet bounds for C_* imply that $L_{C_*,k}$ is countable. Let $L = \bigcup_{k\in\mathbb{N}} L_{C_*,k}$, so that L is still countable. By Lemma 3, choose $p \in [1/2, 1]$ so that for every $\ell \in L$, $\ell(p) \ne \omega(p)$. Then, $C_{p,1/2}$ does not reduce to C_*, because if there were a k-query (zero-error) reduction for some k, Lemma 2 would imply that there was some $\ell \in L$ with $\ell(p) = \omega(p)$. ■

To prove Theorem 2, we need a quantitative lower bound on the error of any approximation of ω by linear functions.

Lemma 4: Pick $p \in [1/2, 1]$ uniformly at random. Then, for any linear function $\ell: \mathbb{R} \to \mathbb{R}$ and any $\varepsilon > 0$,

$$\Pr[|\ell(p) - \omega(p)| \le \varepsilon] \le O(\sqrt{\varepsilon}). \quad (10)$$

Proof.—Let $I = [1/2, 1]$. We first compute

$$\omega''(x) = \frac{1}{2}\left[2\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}\right]^{-3/2} \ge \frac{1}{2} \quad \text{on } I. \quad (11)$$

Hence, ω is uniformly convex on I.

Without loss of generality we can assume that the graph of ℓ intersects the graph of ω twice (with a point of tangency counted as a double intersection). After all, if $\ell < \omega$ on I, translate ℓ up until the first moment of equality with ω, thus decreasing the pointwise error between ℓ and ω at every $x \in I$. If ℓ is then tangent to ω, we are done. Otherwise, ℓ intersects ω at an end point, so rotate ℓ up about this point until it is tangent to ω (no other intersections occur because ω is uniformly convex). Again, pointwise errors do not increase under this rotation. Similar considerations hold if initially $\ell > \omega$ or ℓ intersects ω at one point.

Therefore, suppose ℓ linearly interpolates ω at the (potentially coincident) points $x_1, x_2 \in I$. By a standard argument in interpolation theory, for all $x \in I$, there exists $\xi_x \in I$ such that

$$\omega(x) - \ell(x) = \frac{\omega''(\xi_x)}{2}(x - x_1)(x - x_2). \tag{12}$$

By Eq. (11), $|\omega(x) - \ell(x)| \geq \frac{1}{4}|x - x_1||x - x_2|$. In particular, when $\min\{|x - x_1|, |x - x_2|\} > 2\sqrt{\varepsilon}$, $|\omega(x) - \ell(x)| > \varepsilon$. The probability that p is within $2\sqrt{\varepsilon}$ of either x_1 or x_2 is $O(\sqrt{\varepsilon})$. ∎

Proof of Theorem 2.—Fix $C_*: X \times Y \to A \times B$, where X, Y, A, B are finite and $C_* \in \mathbf{Q}$. Let $L_{C_*,k}$ be the set of linear functions given by Lemma 2. Pick $p \in [1/2, 1]$ uniformly at random. By Lemma 4 and the union bound, for any $\varepsilon_k > 0$, the probability that some $\ell \in L_{C_*,k}$ satisfies $|\ell(p) - \omega(p)| \leq \varepsilon_k$ is at most $O(\sqrt{\varepsilon_k}|L_{C_*,k}|)$. Therefore, by the union bound over k, the probability that there is such an ℓ for any k is

$$O\left(\sum_{k=1}^{\infty} \sqrt{\varepsilon_k}\, |L_{C_*,k}|\right). \tag{13}$$

Choose ε_k so that $\sqrt{\varepsilon_k}|L_{C_*,k}| = c/k^2$, where c is a sufficiently small constant so that the bound in Eq. (13) is strictly less than 1. (Such a c exists because $\sum_k 1/k^2$ converges.) This implies $\varepsilon_k \geq \Omega(k^{-4}|L_{C_*,k}|^{-2})$. Hence, by Lemma 2,

$$k^4 \times (2|X|)^{4|A|^k} \times (2|Y|)^{4|B|^k} \geq \Omega(1/\varepsilon_k). \tag{14}$$

By our choice of ε_k, there exists some p so that for every k, for every $\ell \in L_{C_*,k}$, $|\ell(p) - \omega(p)| > \varepsilon_k$. Choose $C = C_{p,1/2}$. By Lemma 2, if there is a k-query ε-error reduction from C to C_*, then $\varepsilon > \varepsilon_k$. ∎

We end this section with two simple consequences of our main results. We say that $C_1 \leq C_2$ if there is a reduction from C_1 to C_2. We say that $C_1 < C_2$ ("simulating C_1 is strictly easier than simulating C_2") if $C_1 \leq C_2$ and $C_2 \not\leq C_1$.

Theorem 3: For any finite-alphabet correlation box $C_1 \in \mathbf{B}$, there is another finite-alphabet correlation box $C_2 \in \mathbf{B}$ such that $C_1 < C_2$.

Proof.—By Theorem 1, there is a correlation box $C \in \mathbf{B}$ with binary alphabets such that $C \not\leq C_1$. Write $C_1: X_1 \times Y_1 \to A_1 \times B_1$. By relabeling if necessary, we can assume that $0, 1 \notin X_1, Y_1$. Define $X_2 = X_1 \cup \{0, 1\}$, $Y_2 = Y_1 \cup \{0, 1\}$, $A_2 = A_1 \cup \{0, 1\}$, $B_2 = B_1 \cup \{0, 1\}$. Define $C_2: X_2 \times Y_2 \to A_2 \times B_2$ by the following $\mathbf{B}$ protocol. If $x \in X_1$, then Alice does what she would have done in the protocol witnessing $C_1 \in \mathbf{B}$. Otherwise, if $x \in \{0, 1\}$, she does what she would have done in the protocol witnessing $C \in \mathbf{B}$. Bob acts similarly. By construction, $C_1 \leq C_2$ and $C \leq C_2$, so by transitivity, $C_1 < C_2$. ∎

The same technique used to prove Theorem 3 can also be used to generalize Theorem 1 as follows. Suppose we have a finite set of correlation boxes $\{C_*^{(1)}, C_*^{(2)}, \ldots, C_*^{(t)}\} \subseteq \mathbf{Q}$, each with countable input alphabets and finite output alphabets. As in the proof of Theorem 3, we can construct a single correlation box $C_* \in \mathbf{Q}$ with countable input alphabets and finite output alphabets such that every $C_*^{(i)}$ reduces to C_*. Hence, by Theorem 1, there is a correlation box $C: \{0, 1\} \times \{0, 1\} \to \{0, 1\} \times \{0, 1\}$ such that $C \in \mathbf{B}$, but it is impossible to simulate C using shared randomness and any finite number of copies of each $C_*^{(i)}$.

Positive results.—We now show how to construct a finite-alphabet correlation box that is approximately complete for $\mathbf{B}$. The construction is simple, and essentially consists of a discretization of the Bloch sphere [25].

Theorem 4: For every $\varepsilon > 0$, there exists $C_*: \{1, \ldots, T\} \times \{1, \ldots, T\} \to \{1, -1\} \times \{1, -1\}$ with $T \leq O(1/\varepsilon^2)$ such that $C_* \in \mathbf{B}$, and for every $C \in \mathbf{B}$, there is a one-query ε-error reduction from C to C_*.

Proof.—Let S^2 denote the unit sphere in $\mathbb{R}^3$. A pair of local projective measurements as in the definition of $\mathbf{B}$ can be described by $\vec{x} \in S^2$ chosen by Alice and $\vec{y} \in S^2$ chosen by Bob; the (± 1)-valued outcomes a, b satisfy $E[a] = E[b] = 0$ and $E[ab] = \vec{x} \cdot \vec{y}$. (For example, $\vec{x}$ and $\vec{y}$ might specify spin axes along which Alice and Bob are measuring).

Let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_T \in S^2$ be points such that every point in S^2 is within ε of some $\vec{c}_i$ in ℓ_2 distance. Such a collection of points exists with $T \leq O(1/\varepsilon^2)$. The correlation box C_* simply makes the measurements described by $\vec{c}_i$ and $\vec{c}_j$, where i is the input to Alice and j is the input to Bob. By construction, $C_* \in \mathbf{B}$.

For the reduction, when Alice and Bob need to measure according to $\vec{x}, \vec{y} \in S^2$, they simulate the measurement by inputting i, j to C_*, where i and j minimize $\|\vec{c}_i - \vec{x}\|_2$ and $\|\vec{c}_j - \vec{y}\|_2$. Correctness follows from the fact that

$$|\vec{x} \cdot \vec{y} - \vec{c}_i \cdot \vec{c}_j| \leq |\vec{x} \cdot \vec{y} - \vec{c}_i \cdot \vec{y}| + |\vec{c}_i \cdot \vec{y} - \vec{c}_i \cdot \vec{c}_j| \tag{15}$$

$$\leq \|\vec{x} - \vec{c}_i\|_2 + \|\vec{y} - \vec{c}_j\|_2. \tag{16}$$

∎

Proposition 1: There exists $C_*: \mathbb{N} \times \mathbb{N} \to \{0,1\} \times \{0,1\}$ such that $C_* \in \mathbf{B}$, and for every $C \in \mathbf{B}$, $\varepsilon > 0$, there is a one-query ε-error reduction from C to C_*.

Proof sketch.—Use a countable dense subset of S^2. ■

Conclusion.—To better understand quantum entanglement, it is desirable to characterize what can and cannot be achieved using quantum nonlocality. In this Letter, we have ruled out a natural type of characterization, even if attention is restricted to projective measurements of a single Bell pair. We hope that this Letter inspires future researchers to circumvent our results by formulating a different type of characterization of the power of quantum nonlocality.

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